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An Investigation of the Earth-current on the Volcano Aso

Part IV. On the so-called Residuals in the Daily Variation of the Vertical Earth-current at Aso, and the Details of the Observation Arrangements

By Munetosi Namba

(Received June 6, 1940)

Abstract

In the preceding paper, it has been pointed out that the daily variation of the vertical component of the potential gradient of the earth current observed by a dry-well method, contained a remarkable amount of residuals independent of the daily variation of its horizontal components. The present paper deals with the comparison of these residuals with the daily variation of the vertical earth-current observed by a valley-wall method. In addition to the above, the writer will describe the details of the obervation apparatus.

I. Introductory

In the preceding paper, it has been clearly shown that the daily variation of the vertical component of the earth-current observed by an ordinary drywell method in the ground of the Volcanic Laboratory of Aso contained socalled residuals of considerable magnitude independent of the daily variation of its horizontal components.

The possible role of the influence of the variation of the earth-surface temperature as one of the causes of such residuals has also been discussed, and the effect of the atmospheric pressure on it is also pointed out.

The dry-well at the laboratory mentioned in the preceding paper had a depth of 20 meters and a diameter of 2 meters, and two electrodes had been buried 2 meters deep : one at its bottom and the other near the earth surface. Unfortunately, the results obtained with such an arrangement must have been influenced by the difference in temperture between the two electrodes themselves. Again the influence of the local earth-current needed to be taken into account. The writer therefore made further observations by alternative methods. The valley formed by a group of the crater of Mt. Nakadaké and the Toti-

noki Valley offered a very suitable place for the purpose. Arrangements were made at these places to take the records of the vertical earth-current. The comparison of these records with those obtained at the dry-well of the Laboratory revealed very interesting facts which will be presented here.

II. Arrangement at the Crater of Mt. Nakadaké

The deep volcanic canal of Mt. Nakadaké of Aso measures about 1000 meters by 350 meters, with its longitudinal axis north and south. The 1st, 2nd, 3rd, 4th and 5th craters numbered from north to south lie in a row at its bottom. The whole neighbouring region is sandy land, barren because of the volcanic activity. The western side of the inner wall of the first crater forms a precipice about 70 meters in height. On this precipice, two places were selected 50 meters apart in vertical distance, and an electrode was settled in the earth 2 meters or more deep at each place. From each electrode an aerial cable was led to the inner room of a cave-type observatory of concrete str-

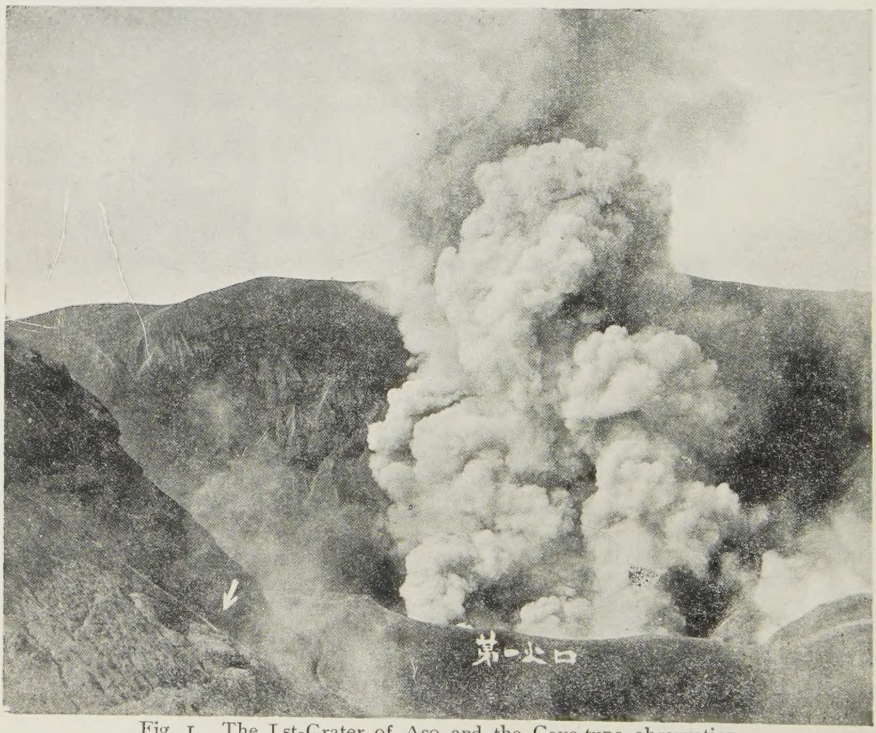


Fig. 1. The 1st-Crater of Aso and the Cave-type observation room (Photo on Sept 5th, 1939)

ucture lying half-way up the crater wall (Fig. 1). This observatory measures 2 meters by 6 meters in dimension and consists of two rooms. The inner dark room above mentioned was equipped with the requisite recording apparatus. The front room served for preparatory work and also for refuge in time of eruption. Fortunately the hummidity in the inner room was so low that there was no anxiety about electric leakage from any part of the apparatus. But unexpected high temperatures, 62°–63°C., during the observation periods and the invasion of volcanic smoke necessitated some special precautions.

(1) To get rid of the troubles caused by the volcanic smoke that might creep in through the doorway, the following precautions were taken.

a. An all-sealed type galvanometer has been newly designed.
   Critical resistances.....1740 Ohms
   Sensibility .....1.8×10^{-6} volts
   Period .....5.5 sec.

b. Also an all-sealed type resistance box made of hard porcelain was used with every metallic part of its terminals sealed in.

c. Every piece of apparatus including the galvanometer, the resistance coil and a watch was concealed in a double wooden box and all the crevices packed with water-proof material.

(2) The wall of the inner dark room was painted elaborately with waterproof cement to prevent moisture and gases from leaking in.

(3) Room temperature was considerably higher than expected before the construction of the room, but financial conditions prevented its reconstruction. The main troubles caused by high temperatures were occasional damage to lamp sockets or of leads, but these were later obviated. In winter, though the outdoor temperature sometimes measured as low as −23°C. at the brink of the crater, the high temperature in the observatory made the work there rather comfortable.

(4) Concerning the potentiometer, except the care about the contact with leads, there was no trouble.

(5) The record of the earth temperature at a depth of 2 meters has been taken with a distant thermograph. It was found to be entirely constant throughout the observation periods. Now the temperature at a depth of 20 centimeters is on record on the advice of Prof. T. Nomitsu.

(6) In this cave observatory, there has been no fear of danger from the volcanic ejecta, and only once, was there so severe an eruption of dense volcanic smoke mingled with yona that breathing was difficult and the observation work became hard to continue.

(7) The most frequent trouble with the observation arrangements was concerned with the cables between the electrodes. At first an underground type was adopted, but was found to be damaged by every landslip caused by heavy rain fall, frost throw or snow throw. To avoid the difficulties of finding the damaged place and repairing it, it was afterwards changed to an aerial type. This cable was also cut several times by suicides as the precipice has been a favorite place for the purpose. It was all painted with pitch and kept in a bamboo tube to protect it from the volcanic gas and mechanical injuries.

In short, the main accidents in the measurement at the active crater were (a) those due to the volcanic bombs and gasses at the time of eruption and (b) those due to the severe cold in winter. The high temperature in the observatory was rather an advantage in winter. Happier still was freedom from worry about leakage that is commonly associated with electric current of low intensity.

III. Arrangement at the Totinoki Valley.

The water gathering from the area of the Valley of Nangô, the southern half of the old atrio of Aso, forms a baranco, the River Sirakawa. Immediately on leaving a water-fall, the Ayugaeri-no-taki, it passes through a deep gorge produced by erosion, called the Valley of Totinoki-Tosita. This valley extends 10 kilometers till the river joins the River Kurokawa which comes from the northern half of the atrio. The floor mersures some 150 meters in width. The right side (the eastern side) is surrounded by a perpendicular wall some 100 meters in height and the left side (the western side) by a screen of the somma some 900 meters above the sea-level). The whole valley is densely covered with old trees and bushes, said to be the only primeval forest on Aso. Under these conditions, the sun rises late in the morning and sets early and the earth-surface temperature fluctuates only narrowly. In the perpendicular wall which stands on the erstern side of the valley, two electrodes have been buried 2 meters deep each and 50 meters apart in vertical distance. The cable from them is led to an observatory of concrete construction half-way underground at Totinoki Onsen, where the records of the variation of the vertical earth-current have been taken automatically.

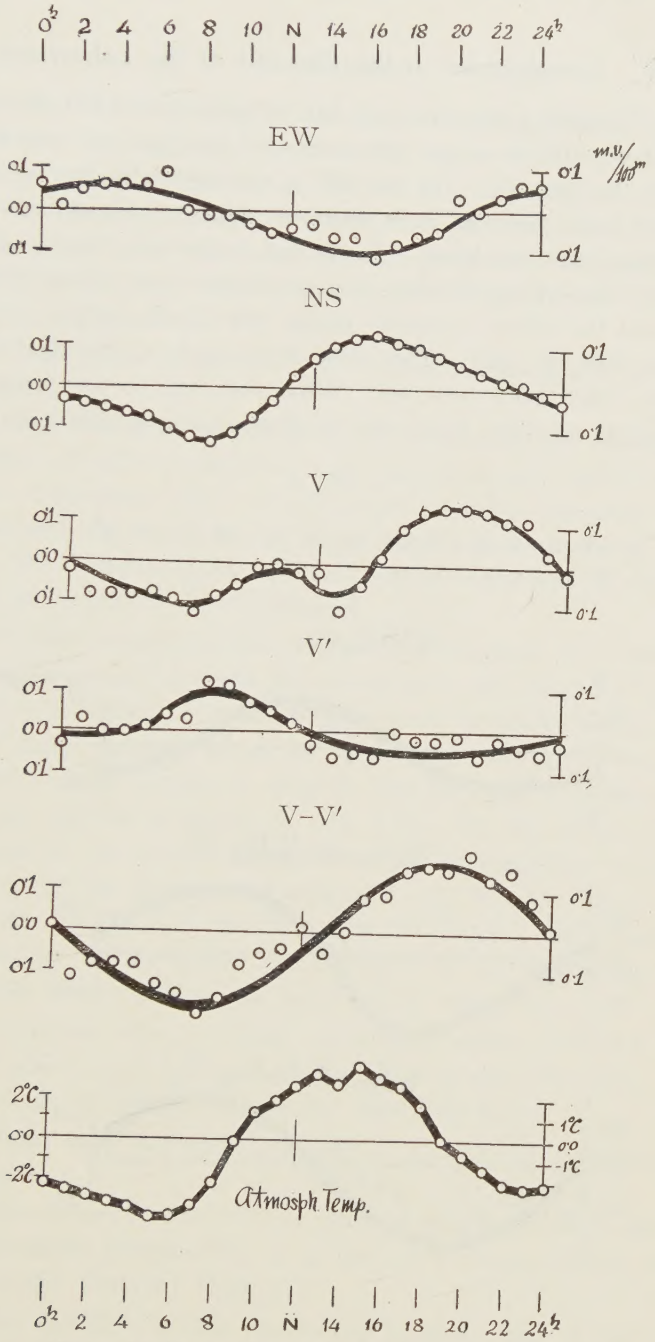
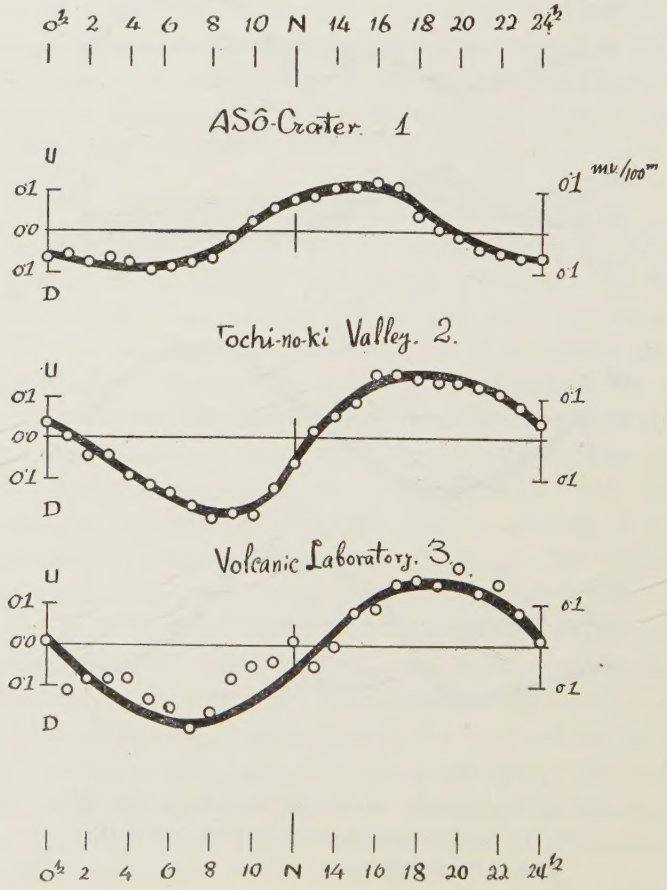


Fig. 2. Daily Variation of the Earthcurrent in the yard of the Laboratory.

IV. Arrangement in the Grounds of the Laboratory.

The Volcanic Laboratory of Aso is located on a hill situated west of Mt. Ebosi, and its main ferro-concrete structure 40 meters square stands on the summit. All the hill is covered with low grasses and consists of deep layer of loam without pebbles. Its altitude is 567,7 meters above the sea level. Three dry wells were dug in the yard. Each well was equipped with two electrodes: one 2 meters below its bottom and the other 2 meters below the earth surface and also 2 meters in from its wall. The cable from each of the electrodes was led to an observatory near by. What concerns to the present paper is the record obtained from one of these wells, which is named V_3.

Fig. 3.



V. Results.

(1) The daily variation of the vertical earth-current at the crater-wall of Aso. The resistance between the upper and lower electrodes meaured 250 ohms at the end of July 1939. A resistance was inserted between them and from the resultant series resistances of 70,000 ohms, 160 ohms have been taken for the purpose of recording. The mean daily variation of the vertical earth-current during 14 days in August 1939, is shown in Fig. 3 (1).

(2) The daily variation of the vertical earth-current at the Totinoki Valley.

The risistance between the electrodes measured 300 ohms. From 30,000 ohms of series resistance, 100 ohms have been taken for measurement. The mean daily variation during 11 days covering the end of August and the beginning of September is shown in Fig, 3 (2).

(3) The daily variation of the vertical component of the earth-current in the yard of the Laboratory.

The resistance between the electrodes measured 200 ohms. resistance was inserted between them and from the resultant series resistance of 9,000 ohms, 600 ohms have been taken for measurement. The mean daily variation of of the vertical and horizontal componets of the earth-current during 11 days in August 1939, is shown in Fig, 2. And the residual is shown in Fig. 3 (3). In this season, the upward change of the vertical component was accompanied by the W-S fluctuation of the horizontal component, and the downward change of the former by the E-N fluctuation of the latter. The results have been treated in the same way as is shown in the preceding paper. The residuals thus obtained are tabulated in the table.

By comparing these residuals with the results obtained at the Crater-wall and at the Totinoki Valley, Fig. 3 was obtained. Close examination of this figure reveals the following facts:

(a) There is a noticeable parallelism among the records of the three localities.

(b) The variation curve in the yard of the Laboratory coincides, though with slight irregularities at 5 and 10 o'clock, almost entirely with that of the Totinoki Valley.

(c) The variation at the crater-wall is about half of the others in magnitude.

(d) The minima (the extremes of the downward change) occur

earlrest at the crater-wall, next in the yard of the Laboratory, and latest at the Totinoki Valley.

(c) The maxima (the extremes of the upward changes) occur earliest at the crater-wall, next at the Totinoki Valley, and latest in the yard of the Laboratory.

How can these facts be interpreted? The writer's view will be presented in the next section.

	$E_W \frac{\text{m.v.}}{100^m}$	$NS \frac{\text{m.v.}}{100^m}$	$EW + NS$	V	$\frac{Ew+Ns}{V}$	$\frac{Ew+Ns}{1,1} \equiv V'$	$V - V'$	V in the Crater	V in the Totinoki Valley	Atmospherci Temperature
0h	0.06	−0.03	−0.03	−0.02	1.5	−0.03	0.01	−0.06	0.04	−2.2
1h	0.01	−0.04	0.03	−0.08	0.4	0.03	−0.11	−0.05	0.01	−2.5
2h	0.05	−0.05	0.0	−0.08	0	0.0	−0.08	−0.07	−0.04	−2.8
3h	0.06	−0.06	0.0	−0.08	0	0.0	−0.08	−0.06	−0.04	−3.1
4h	0.06	−0.07	0.01	−0.07	0.1	0.01	−0.08	−0.07	−0.09	−3.3
5h	0.06	−0.10	0.04	−0.09	0.4	0.04	−0.13	−0.09	−0.11	−3.8
6h	0.09	−0.12	0.03	−0.12	0.3	0.03	−0.15	−0.08	−0.13	−3.7
7h	0.0	−0.13	0.13	−0.08	1.6	0.12	−0.20	−0.07	−0.16	−3.2
8h	−0.01	−0.11	0.12	−0.05	2.4	0.11	−0.16	−0.06	−0.19	−2.1
9h	−0.01	−0.07	0.08	−0.01	8.0	0.07	−0.08	−0.01	−0.18	−0.1
10h	−0.03	−0.03	0.06	0.0	—	0.05	−0.08	0.03	−0.18	1.3
11h	−0.05	0.03	0.02	−0.02	1.0	0.02	−0.04	0.06	−0.12	1.6
12h	−0.04	0.07	−0.03	−0.02	1.5	−0.03	0.01	0.08	−0.06	2.6
13h	−0.03	0.10	−0.07	−0.11	0.6	−0.06	−0.05	0.09	0.02	3.2
14h	−0.06	0.12	−0.06	−0.05	1.2	−0.05	0.0	0.11	0.06	2.7
15h	−0.06	0.13	−0.07	0.02	3.5	−0.06	0.08	0.11	0.09	3.6
16h	−0.11	0.11	0.0	0.09	0	0.0	0.09	0.12	0.16	3.0
17h	−0.08	0.10	−0.02	0.13	0.2	−0.02	0.15	0.11	0.16	2.6
18h	−0.06	0.08	−0.02	0.14	0.1	−0.02	0.16	0.04	0.15	1.7
19h	−0.05	0.06	−0.01	0.14	0.1	−0.01	0.15	0.01	0.14	0.0
20h	0.03	0.04	−0.07	0.13	0.5	−0.06	0.19	−0.01	0.14	−0.7
21h	0.0	0.02	−0.02	0.11	0.2	−0.02	0.13	−0.04	0.13	−1.4
22h	0.03	0.01	−0.04	0.11	0.4	−0.04	0.15	−0.05	0.11	−2.1
23h	0.06	−0.01	−0.05	0.03	1.7	−0.05	0.08	−0.06	0.08	−2.3
24h	0.06	−0.03	−0.03	−0.02	1.5	−0.03	0.01	−0.06	0.04	−2.2
	Mean of 11 days in Aug. 1939				Mean 1.1			Mean of 14 days in Aug. 1939	Mean of 11 days in Aug. - Sept.	Mean 23.°5C Mean of 11 days in Aug. - Sept.

VI. Discussion

As has been stated, the residuals obtained from the daily variation of the vertical component of the earth-current observed by the dry-well method in the yard of the Laboratory resemble closely the daily variation of the vertical earth-current obtained by the valley-wall method at the Totinoki Valley. What may be the meaning of such resemblance?

The writer's view runs thus: The apparent variation of the vertical earthcurrent observed at any locality is composed of the inherent variation proper to or characteristic of that locality at the time of observation and the secondary variations conditioned by the variations of its horizontal components or by any other factor correlated with it. Such secondary variation may be called for the present, an accessory variation. The same reasoning may apply to the daily variation. In the case of the dry-well method, the apparent daily variation should be to a considerable degree a subject of this accessory variation since the dry-well method is itself subject to criticism as has been noted in the introduction of this paper. The so-called residuals obtained with consideration for the daily variation of its horizontal components may be recoguised as manifesting mainly the inherent variation. On the other hand, in the valley method, with the greater distance between two electrodes, the influence of the accessory variation must be smaller. Therefore, in this case the daily variation of the vertical earth-current may be assumed to have been governed mainly by its inherent variation. This is why the residuals obtained by the first method coincide with the daily variation itself observed by the second method. The writer is sorry to have on arrangement at present at Totinoki Valley to ascertain the magnitude of the accessory daily variation, to correct the daily variation of the vertical earth-current and thus to measure the real inherent daily variation of the earth-current at that locality. But the following facts may complement the deficiency to some extent:

(1) As is shown in Fig. 4, the record of the variation of the vertical component in the Totinoki Valley shows also the irregularities at 5 and 10 o'clock corresponding to those in the residuals of the daily variation in the dry-well method.

(2) If we compare the record of the daily variation of the verti-

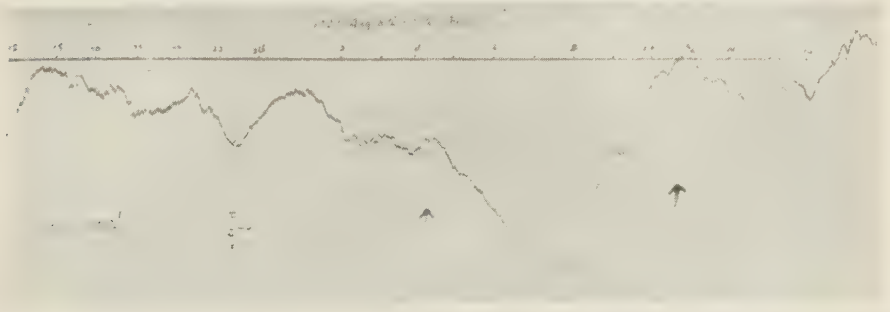
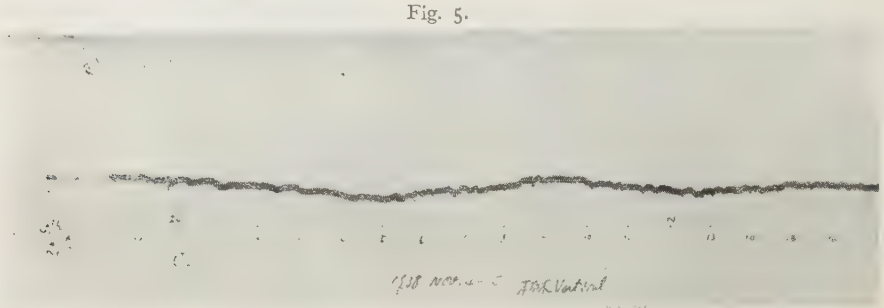


Fig. 4. Record at Totinoki (8–9 Aug. 1939).

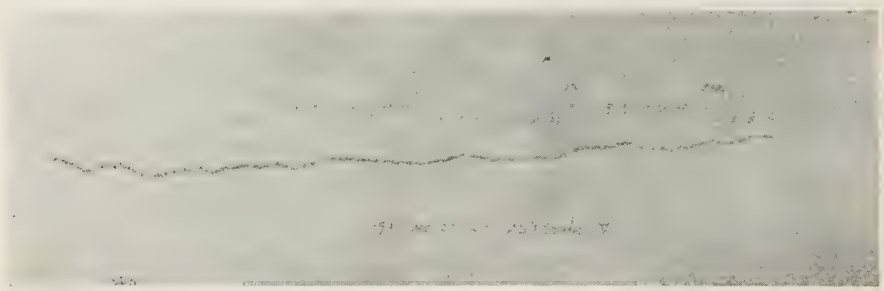
cal component at the Totinoki Valley taken in November 1938, Fig. 5, with the accessory variation in the yard of the Laboratory, V', Fig. 2, they coincide very closely.

(3) Similarly, the record at the crater taken in December 1938, Fig. 5, also resembles the latter (V'). Fig. 6 represents a record of the valley-method.

Fig. 5.



Record at Totinoki-Valley. (1938 Nov. 4-5).



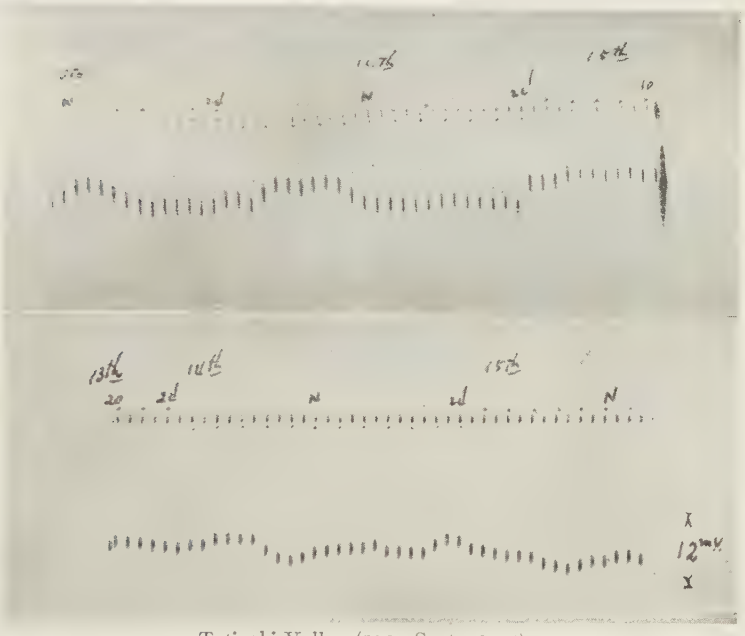
Record at the Crater. (1938 Dec. 23-28).

Then comes the question what is the cause or causes of the inherent or characteristic daily variation of the vertical component. The writer has discussed in the previous paper, the possible role of the daily variation of the earth-surface temperature as at least one of its causes and has made a suggestion of the effect of the variation of the atmospheric pressure. Whether the daily variation of the earth-surface temperature is its sole cause or not cannot be decided at the present state of our knowledge, but it must be admitted to be at least a dominant factor judging from the following facts:

(1) In connection with the sunshine, the above three curves can be interpreted as follows: The minimum values (that is the largest fluctuation toward Down) appear earliest at the sandy crater (curve 1), next in the grassy yard of the Laboratory (curve 3), and latest at the Totinoki Valley with its dense plantation (curve 2).

(2) The maximum values (that is the largest fluctuation toward Up) appear in the order of Curve 1, Curve 2, and Curve 3. As previously stated, the west side of the Totinoki Valley is bounded by a somma wall measuring 900 meters above the sea level and there the sun sets earlier than at the yard of the Laboratory.

Fig. 6.



Totinoki-Valley (1939 Sept 13-15).
Crater (1939 Sept 13-15).
Records of the Valley method

(3) The yard of the Laboratory and the Totinoki Valley are situated on the so-called primary lava plateau formed by Mt. Ebosi-daké and the homogeneous loamy land at these places belongs to the most ancient in the atrio of Aso. But the crater is very porous ashy land newly formed. Accordingly there is very distinct difference between them in the moisture content and we may expect to find considerable difference in the amplitude between daily variations of the earth-current at these localities.

(4) The amplitude of the daily variation is not unreasonably large, judging from the results reported in the previous paper.

From these points of view, it seems to be permissible to assert that the characteristic daily variation of the vertical component of the earth-current is caused by the daily variation of the earth-surface tem-

perature and other factors connected with it. The result reported by Walker on the variation of the earth-current between the foot and the summit of a mountain can be regarded as the characteristic daily variation of the vertical component, though the considerably great horizontal distance between two electrodes is a matter for criticism.

Lastly, there remains a question about the most suitable of method for observing the variation of the vertical component of the earth-current. Consideration of the following points may furnish useful ground for its decision:

(1) The Dry-well Method.

(a) Generally, it is difficult to make a deep dry-well and obtain a long distance between two electrodes. Accordingly results obtained by this method are apt to be influenced considerably by the accessory variation.

(b) The great difference between the variations in temperature of two electrodes themselves is a source of apprehension.

(c) But it has the advantage that it can be placed at any place desired.

(2) The Valley-wall Method.

It has the following characteristics.

(a) In general, as the long distance between two electrodes can be obtained whenever a suitable valley-wall is available, the influence of the accessory variation of the vertical component can be minimized. Moreover, the influence of the stray current due to the leakage from an electric track can be avoided.

(b) It is an advantage of this method that the temperature of the two electrodes can be maintained almost alike.

(c) But unfortunately, it is impossible to obtain at any desired locality a valley suitable to set up the measuring equipment.

Though each method has its merits and its weak points, the valley-wall method is desirable for the study of the characteristic variation of the vertical component whenever it is practicable. The writer adopted this method for the study of the relationship between the volcanic activity of Aso and the vertical earth-current. The study of Palmieri at Mt. Vesuvius on the variation of the earth-current can be classed as a study of the variation of the vertical earth-current by the valley-wall method though not in a strict sense.

This study has been accomplished under the continued guidance of Prof. T. Nomitu and under the financial support of the Japanese Society for the Promotion of Scientific Research. The writer wishes to take this opportunity to express his thanks.

An Investigation of the Earth-current on the Volcano Aso

Part V. Correlation of the Volcanic Activity of Mt. Aso with the Variation of the Vertical Earth-current

By Munetosi Namba

(Received May 18, 1940)

Abstract

This is the summary report of a study on the variation of the vertical potential gradient of the earth-current of Aso, with special reference to its relation to the sequence of the volcanic activity. The relation revealed is interpreted with the experimentally proved fact that the positive local pressure constitutes a center of the negative electric potential. Also the problem of forecasting the volcanic activity is discussed.

I. Introductory

The occurrence of nearly constant upward earth-current in the surface layer of a slope is an established fact of early discovery. The writer made a survey on the distribution of the potential gradient of the upward earth-current at the flank of the cinder cone "Kométuka", the top of which is 954 meters above the sea-level. For companison, he also performed an experiment with a soil column, following the method adopted by S. J. Mauchly. As the result of these studies, it has been clearly shown that the application of a positive pressure upon a part of homogeneous soil results in the fall of the electric potential in the electrode nearer to the pressure center and its rise in the remote electrode; that the magnitude of the potential difference thus produced is directly proportional to the intensity of the pressure applied, and that negative pressure produces the reverse effect. These statements can be simplified thus: A positive local pressure constitutes a center of negative electric potential.

If this is true, the high pressure produced far inside a crater by the accumulation of gas previous to an explosion must cause the change of the potential gradient of the earth-current. Though the horizontal gradient of the earth-current is affected to a great extent by the ionization conditions in the upper layer of the atmosphere and also by the so-called stray current in the surface of the ground, the vertical potential gradient is relatively unaffected by such external

factors. The writer, therefore, tried to draw a parallel between the volcanic activity and the variation of the vertical potential gradient of the earth-current.

There are two methods of measuring the variation of the vertical potential gradient of earth-current, namely the dry-well method and valley-wall method. In the latter, any suitable natural valley with a perpendicular cliff is utilized and electrodes are buried in the cliff. In the former, a dry-well is dug as deep as possible into the earth's crust and two electrodes are inserted near the top and the bottom of it. The characteristics and relative advantages of these methods have already been discussed. The method adopted in this case was the valley-wall method, the western inner wall of the first crater of Mt. Nakadaké being utilized. A detailed description concerning the whole arrangement at Mt. Makadaké and special care about the measuring apparatus has beed given in the preceding paper.

From a close examination of the daily variation of the vertical gradient of the earth-current, it has been found that the rise of the earth-surface temperature causes the eqpansion of the earth-surface texture, which exerts various kinds of influences on the vertical component. Also it has been directly observed that rapid rise in earth-surface temperature exerts distinct downward pressure. And the effect of the atmospheric pressure on the vertical component was also pointed out. Thus we have arrived at the belief that the influence of the rapid change of the earth-surface temperature will play a more important part in the change of pressure within the earth crust than that of the variation of atmospheric pressure on the variation of the vertical potential gradient of the earth-current. Moreover, the existence of a satisfactory parallelism has been found between the daily variation of the earth-surface temperature and the amount of residuals, which signifies mainly the inherent or characteristic variation of the vertical potential gradient of the earth-current, obtained when the apparent daily variation of the vertical component is compared with that of the horizontal earth-current.

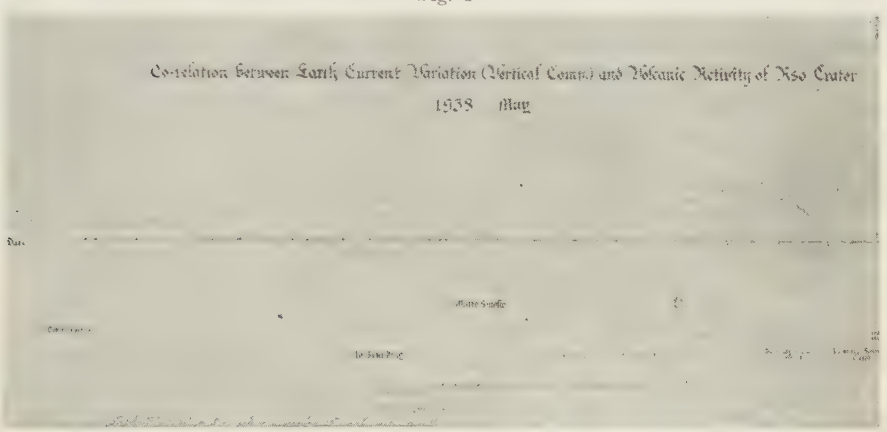
Under the guidance and encouragement of Prof. Nomitsu and by the support of the Japan Society for the Promotion of Scientific Research, the writer was able to carry out the observation on the variation of the vertical potential gradient of the earth-current at the first crater of Mt. Nakadaké of Aso in 1938–1939. This is the summary report of the results accumulated during these two years.

II. Results of Observation on the Volcanic Activity and on the Variation of the Vertical Earth-current at Aso

The object of the present study was, as before stated. the first crater of Mt. Nakadaké, the most active crater of Aso. Observation began on the 25th of December, 1937, and closed at the end of September, 1939. As to the general view on its activity, though no large explosion occurred, a number of small activities took place continuously during this period. And as the arrangement for the measurement of the vertical potential gradient was set up within the crater, which is near the center of its activity, there was no hindrance to learning the general features of its behavior. Some representative cases will be presented here with the necessary explanation.

(1) The Activity about the Middle of May, 1938.

Fig. 1



This is the second activity of the first crater after the commencement of observation. As the writer was then staying at the cave-type observatory in the crater-wall to keep watch, a nearly complete record was obtained. Some of the results have already been reported at the General Meeting of the Japanese Association for the Advancement of Science held in October, 1938. Fig. 1 is a transcription of the record on the variation of the vertical potential gradient of the earth-current during the period from the 1st to the 18th of May. The ordinate indicates the variation of the vertical component per 100 meters in m. v. and the rise of the curve indicates the 'up' change, namely the increase of the potential gradient of the upward compo-

nent. In the phrase we have adopted, the increase of the earth-current or the rise of the potential of the upper electrode is called a change toward 'up' or an up change, and the reverse case a change toward 'down' or a down change. The abscissa indicates the time, each record having heen made at noon of the date given. For the sake of comparison, a standard line is drawn in the figure based upon the value about the 9th of May.

General examination of the figure will reveal the following prominent facts: The vertical gradient, which continued almost constant during the early part of the month, turned toward 'up' on the 9th of May. It rose impetuously afterwards to reach the maximum value noted from the 12th to the 15th, exceeding the standard line by some 100 m. v. And then it turned towards 'down' and settled down to the neighborhood of the standard line.

As to the activity of the first crater, after remaining quiet during the early part of the month, the eruption of white smoke increased from about the 8th onward. Though the silent state prevailed, a momentary but violent eruption of white smoke was observed at intervals. On the 11th day the crater was in a quiet state. It became more active in the morning of the 12th, and pulsatory rumbles of explosive nature though on a small scale, continued from the 12th to the 15th. After that time, though a vomiting of some smoke with a vestige of eruption still lasted, the pulsatory rumbling faded away rapidly.

During the two days of the 13th and 14th, the pulsatory rumble reached a climax, and the well-known flashes observed at the time of the activity in February, 1933, were again distinctly observed. As many as 3 or 4 flash rings were counted. The view at night was splendid.

As a rule, when such a pulsatory rumble dominates, the ejection of ash or white smoke becomes very feeble and the detonation and ejection of red-hot lava fragments becomes the leading part of the volcanic activity. The pulsatory rumble is usually repeated almost rhythmically. Though its interval varies from several seconds to several minutes, it is generally short at the beginning of the activity, growing longer and longer till it lasts as much as 10 minutes or more. And finally it changes to a continuous roar of undertone. The rumble is accompanied by the dispersion of red-hot lava fragments. After the activity reaches the the stage where the two states, namely a

pulsatory rumble accompanied by the eruption of lava fragments and roar of undertone accompanied by an ejection of a great amount of ash, alternate, the rumble fades away rapidly. In our present case, the rumble alternated at times with a roar of undertone and an ejection of a considerable amount of ash from the midnight of the 13th onwards. It ceased entirely on the 16th. The ejection of smoke lasted for some time longer, suggesting dying embers, until at last it also ceased altogether.

At the time of the pulsatory rumble, a number of volcanic thunders and volcanic showers were experienced. The volcanic thunders sounded only as unearthly snaps, as they were experienced near by. The showers burst violently and fell like a belt to the zone where the volcanic smoke hung. The lightning seemed as if emitted from the cluster of volcanic smoke. It is likely to appear when the atmosphere gives signs of a shower.

When the variation of the vertical gradient is compared with the sequence of the volcanic activity, the following facts will be found: The pulsatory rumble commenced when the curve of the vertical earth-current turned up and approached its shoulder; the eruption became violent when the curve about reached its maximum or had just passed the maximum; and when the curve turned down, the rumble began to fade away, only the ejection of smoke lasting longer. At last the volcanic activity ceased entirely as the curve settled down to the initial position. In the explanation of Fig. 1, the word 'fragments in white smoke mingled with lava fragments' means not the red-hot lava fragments but fragments that were lying about the mouth of the crater, lifted with the white smoke when it is ejected in great amount. Those which are actually ejected at the time of a pulsatory rumble consist mainly of red-hot lave fragments, or more correctly magma fragments, and they display a splendid sight of fireworks at night.

A close examination of Fig. 1 reveals that the vertical potential gradient at the begining of the month was higher than the standard line and its minimum value (that is the maximum change toward down) appeared about the 7–8th of May. This can be explained as the after effect of the up change at the end of April, its gradual fall tailing over the early part of May. Again, it is noticed that the curve of the vertical potential gradient turned toward up at the end of May to reach its maximum at the beginning of June.

Here let us study an interesting question: The phenomenon of

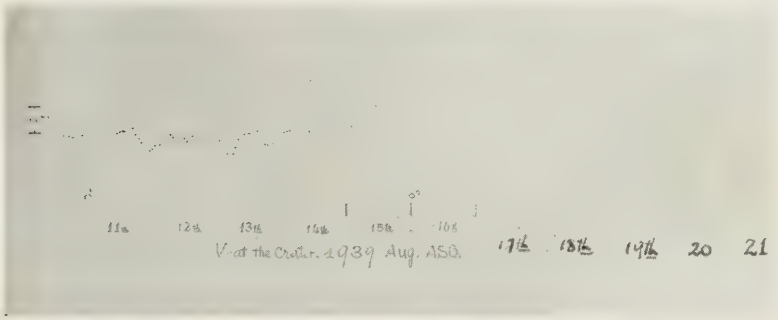
violent explosion attracts general notice, but does this phenomenon occur quite suddenly? Let us re-examine the activity of May. As the activity of Volcano Aso is usually gentle, the commencement of the pulsatory rumble and the ejection of the red-hot lava fragments passes in most cases without public notice, and the activity becomes noticeable from a distance only after it has grown to the stage of violent eruption of smoke such as was observed at 2 a. m. of the 13th. But as this happened in the middle of the night, it was not until the next morning that the activity was fist noticed by distant observers. The usual photograph referred to as the sight of the explosion of Volcano Aso is one taken after its activity had reached this stage. Especially the photographs of eruptions of Mt. Aso in early times when communication was difficult and the reports of distant observers, must be regarded as referring almost entirely to this stage. In our case, though the surface activity could be distinctly observed at the scene, on the 12th, a distant observer must have noticed it for the first time on the 13th, a day after its actual commencement. The violent activity of 1933 is generally reported to have begun on the 24th of February, but it is significant that the observer, unable to stay any longer at the brink of the crater, descended at midnight of the 23rd.

Thus, if the time of commencement of the surface eruption is defined as that of a pulsatory rumbling, it coincides with the time when the curve of the vertical potential gradient reaches its shoulder and the eruption becomes violent when or just after the curve reaches its maximum. Alternatively, if it is defined as the time of the commencement of violent eruption, then it coincides with the time when the curve reaches its maximum or just after it. If we remember that in our case the outward destructive action did not become so violent as to be noticeable to everyone till the 13th of May, we may adopt the latter definition. At any rate, it must be noticeable that the eruption of Volcano Aso does not happen without warning, but proceeds by stages, and that the record of its eruption must be different according to the site of observation. The writer believes that any previous statement about Aso that the eruption broke out "suddenly with a terrific rumbling" does not describe the real precess of its activity. In other words, we may reasonably assume that in previous eruptions of Aso the surface eruption had already commenced about one day before the time reported.

Activity similar in duration to the one described in May, 1938, was also experienced in April, 1938; in April, 1939; and in the period covering the end of September and the beginning of October, 1939. All of them lasted for several days. It is a hard and laborious task to take complete records in such cases.

(2) The Activity about the Middle of August, 1939.

Fig. 2



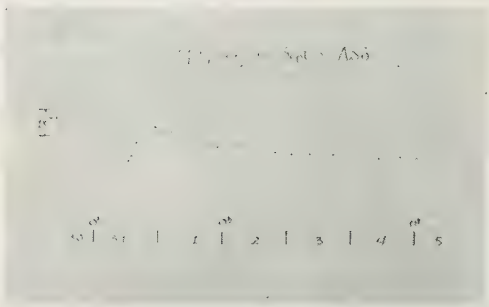
In August, 1939, the ash-fall continued almost every day from the beginning of the month. It was rather severe, and as the east wind prevailed the pasturage in the western half of the old atrio of Aso was almost destroyed. But during those days, the outward appearance of the crater seemed quite quiet and a silent state lasted. Then the pulsatory rumble, though on a small scale, commenced on the 15th and continued till the 18th. The ejection of ash decreased as usual and that of red-hot lava fragments took its place. But the distribution of the latter was limited to the area of the crater itself. At last, the last traces of ash-fall like burned-out embers ceased entirely on the 24th, and a silent state without smoke was restored again.

The record of the variation of the vertical potential gradient shown in Fig. 2, shows that it changed conspicuously toward 'up' on the 15th, and then turned toward 'down' on the 18th, and fell contintously to the mimimum value on the 24th. After that, it changed again towards up to form the next rise, at the end of August and the beginning of September.

(3) The Activity during the Period from the End of August to the Beginning of September, 1939.

After remaining in a silent state with some ash ejection, the first crater became somewhat active on the 31st of August and a gentle pulsatory rumble continued from the 31st of August to the 1st of

Fig. 3



September. The distribution of lava fragments did not extend beyond the crater brink or rim. Meanwhile the ash ejection diminished as usual. While the curve of the variation of the vertical potential gradient turned towards up from the minimum value of the 24th of August onward, as is shown in Fig. 3, and the rate of rise became gradually more rapid. It tended to rise after the beginning of September and exhibited a variation coincident with the variation of the crater activity at the end of September and the beginnig of October.

(4) The Activity during July 23–30, 1938.

The crater, after remaining in a silent state with some ash ejection, raised on the 27th of July a pulsatory rumble that lasted during the next day, while the ash ejection diminished. The lava ejection was but small in amount. On the 29th, the rumble ceased and it returned to the state of continuous ash ejection. The curve of the vertical potential gradient after remaining in a quiet state, as is shown in Fig. 4, turned, rapidly towards up from the 26th onwards, and was restored to the initial state by the 29th.

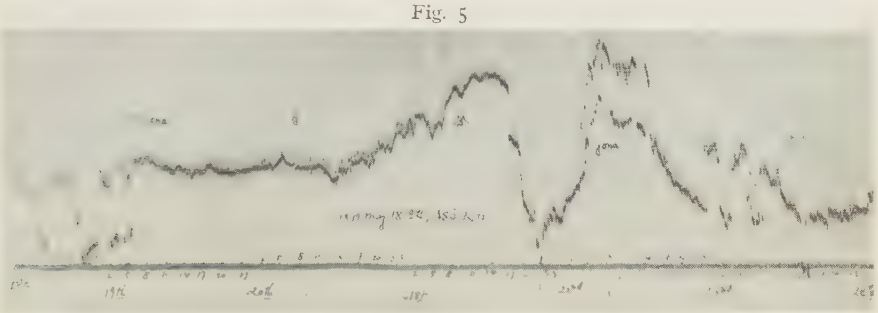
(5) The Activity during May 18–24, 1939.

As is shown in Fig. 5, the record on the variation of the earth-current runs as follows: After remaining continuously almost in a tranquil state, it changed rapidly towards up about the 21st; after a temporary steep change towards

down at the midnight of the same day, it turned again towards up; then it was restored to the initial value from the 23rd onwards.

The sequence of the activity of the first crater was as follows: On the 18th, white smoke; on the 19th and 20th, white smoke mingled with some ash; on the 21st, the violent pulsatory rumble noticeable at the Laboratory 7 kilometers west of the crater. This condition continued also in the morning of the 22nd. On the 23rd, the activity degenerated and became silent, though the ejection of ash hung about, trailing towards the north. On the 24th, the ejection of ash mingled with lapilli continued without rumble. On the 25th, it came to the stage of white smoke without rumble, The ejection of lapilli fell off decidedly.

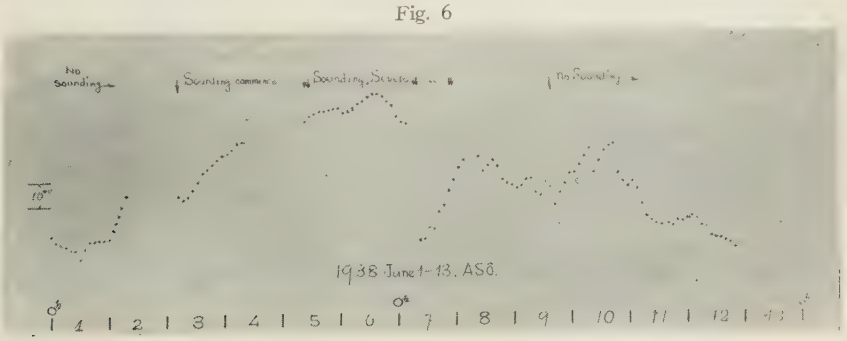
Fig. 5



(6) The Activity from June 1 to 13, 1938.

The variation of the vertical potential gradient of the earth-current, which is shown in Fig. 6, behaved much the same as in the previous case. As to the activity of the first crater, after remaining silent with white smoke the pulsatory rumble commenced on the 3rd of June. The violent pulsatory rumble lasted during a period of five days, from the 5th to 9th; then it faded and reverted to the initial quiet state again. Though the rumble was very violent, the ejection of lapilli and ash was rather moderate in amount.

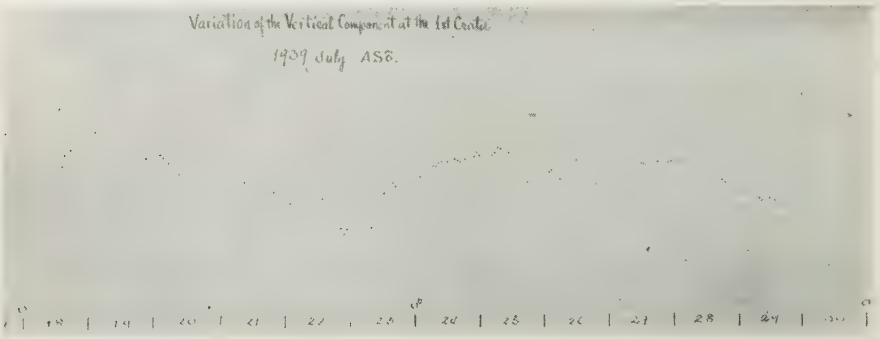
Fig. 6



(7) The Activity in July, 1939.

In this case, the variation of the vertical potential gradient was somewhat complicated as is shown in Fig. 7, and exhibited a composite type as if formed from the first type intertwined with those noticed in the preceding two sections, (5) and (6). As a whole, however, it is a continuous down change mixed with a sharp up change, though on a small scale.

Fig. 7



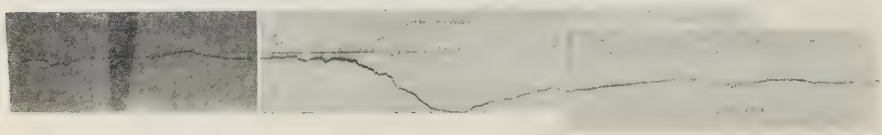
About the crater activity, the long-continued ejection of smoke was suspended for a while from the 23rd of July onwards and a pulsatory rumble commenced on the 25th and lasted till about the midnight of the 25th or the morning of the 26th. Then the silent state accompanied by the ejection of ash was restored.

Such small scale activity was observed so many times during the term of the present study that no record was made of them all.

(8) The Activity during January 15–21, 1939.

At the first crater, the ejection of ash continued every day from the opening of the year, but on the 18th of January, the eruption changed to mist-like white smoke. During the 19th and 20th, a silent and smokeless state lasted, A rumble accompanied by the ejection of lava fragments commenced on the 21st. The continuous ejection of ash began again next day and continued afterwards.

Fig. 8



The behavior of the vertical potential gradient during this period is shown in Fig. 8. This type was also observed repeatedly. Its

characteristic feature is the acute down change with the period of quiet before and after it. But when it is examined more closely, two adjacent active periods will be found: One at the end of December and the beginning of January and the other at the end of January and the beginning of February. In these periods, the roars were so violent as to be heard at the Laboratory. The middle of January corresponds exactly with the transition between these two active periods. So the down change of the vertical potential gradient may be considered as the indication of a cessation of the volcanic activity. When thus viewed, this type is not at variance with other cases.

Above is a brief list of apparently different types in the behavior of the vertical potential gradient with reference to the crater activity. On the examination of these data, some general conclusions may be drawn:

(i) The typical sequence may be represented by the following scheme: The rapid change of the vertical potential gradient of the earth-current towards 'up' is followed directly by the break of a quiet state and the commencement of a pulsatory rumble. And just when or immediately after the former reaches its maximum, the rumble begins to fade and the ejection of ash increases in amount. And finally the initial quiet state is restored.

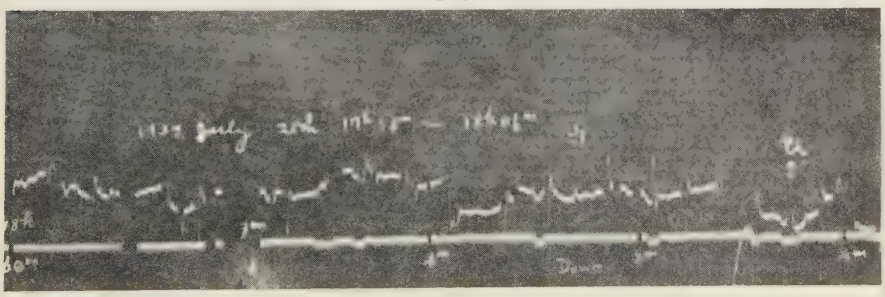
Closer examination will also reveal the following facts:

(ii) As to the behavior of the vertical potential gradient before its rapid rise, there are two cases: One with a frequent fluctuation and the other with a rather smooth course.

(iii) As to its behavior after the volcanic activity, there are also two corresponding cases, namely the descent of the vertical potential gradient with frequent fluctuation and that with smooth process.

(iv) Generally, in the vicinity of the maximum value, the vertical potential gradient varies conspicuously without refereuce to the amplitude.

Fig. 9

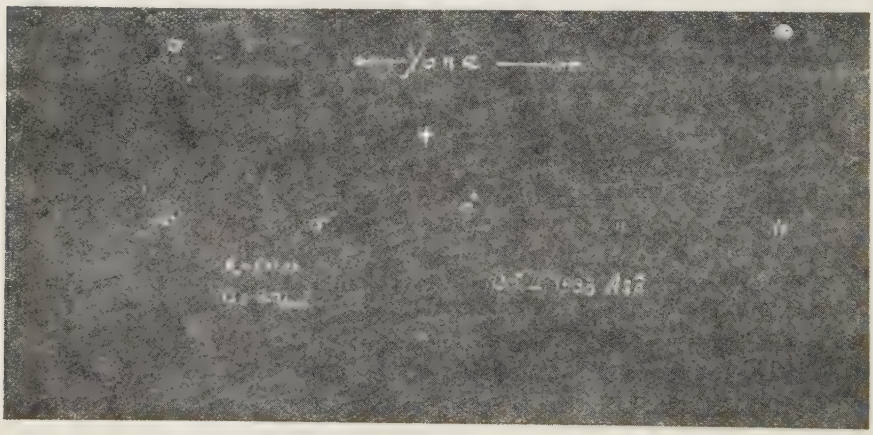


(v) The value of the vertical potential gradient of the earth-current after the activity does not necessarily coincide with its initial value; it may be sometimes higher and sometimes lower than the latter.

Nevertheless, the rapid change of the vertical potential gradient toward 'up' is always accompanied by the commencement of the pulsatory rumble. Moreover, this phemomenon seems to be universal irrespective of the violence of the volcanic activity. In the beginning of the descending stage of the vertical potential gradient the volcanic activity is usually already in process.

L. Palmieri has found that the earth-current between the summit and the foot of Mt. Vesuvius turns towards 'up,' (to use the writer's expression), and finally returns to the initial state as the activity decays and that the daily variation is obscured by this variation. These observed facts agree with the present results on the variation of the vertical potential gradient obtained by the valley-wall method at the first crater of Aso.

Fig. 10



According to E. O. Walker, the earth-current is influenced by atmospheric electricity at Kandy. With the present arrangement, the vertical potential gradient of the earth-cerrent was affected to a considerable degree by the electric discharge of thunder, as seen in Fig. 9, and as is usual also in the record of the horizontal potential gradients of the earth-current. This fact must be remembered in interpreting the data. Also the volcanic smoke of Aso was found to make the atmospheric charge negative, as seen from Fig. 10, a record taken at the Laboratory. But this influence was unnoticeabl in the record

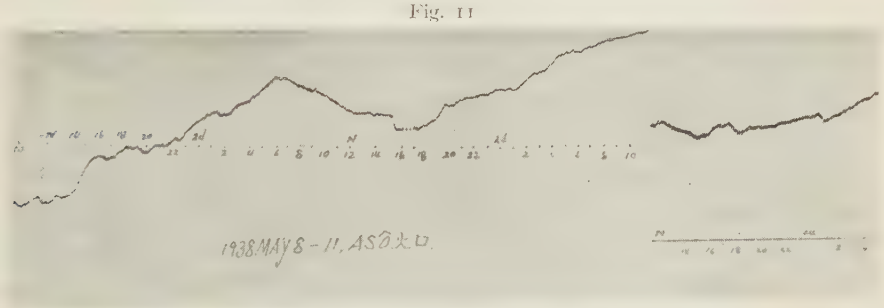
of the crater-wall earth-current of Aso. This may, however, be ascribed to the sensibility of the instrument, and further study is required to check this point. Nevertheless, if we recognize that it has an influence, it would exert a down change, and the above interpretation will cover it.

III. The Problem of Forecasting the Volcanic Activity of Aso

From what has been stated in the preceding chapter, the fact of correlation between the variation of the vertical potential gradient of the earth-current and the activity of the first crater of Aso seems more than probable. Then comes a question: Is it possible to forecast tne volbanic activity from the behavior of the vertical potential gradient of the earth-current? The writer's view deduced from his own experience during this term runs as follows:

(i) As was experienced at the time of activity in May, 1938, if watched with apparatus of high sensibility, the vertical potential gradient will rise so conspicuously that the recording paper must be shifted upwards, indicating the approach of the active period; and the rapidity of rise tells us that the violent activity is within one or two days. Again the down change necessitates changing the recording paper and while the volcanic activity is still in process, appries us of its decay. (Fig. 11).

Fig. 11



(ii) As for the violence of activity, all the volcanic activities experienced during this term were rather moderate and every one of them allowed its complete record to be taken in a single recording paper. Therefore, no remarkable record was obtained. For this purpose, it is desirable to have several sets of apparatus differing in sensibility and to take a set of complete records for every activity.

(iii) Judging from the sequence of the volcanic activity experi-

enced durieg this term, the smaller the rate of the up-change of the vertical potential gradient i. e. the longer it takes to complete the up-change, the greater the volcanic activity will be, and vice versa. So if we judge correctly the composite type, the general features of the coming volcanic activity can be forecast.

(iv) Lastly, the most difficult question of all must be considered whether the above stated behavior of the vertical potential gradient is necessarily followed by a surface volcanic eruption or not. The behavior of the vertical potential gradient and the volcanic surface activity coincided in all cases experienced during this term, and there was no doubtful case. But volcanic activity comprises also phenomena other than the surface activity. There must presumably be a case where, though the vertical earth-current may behave typically, the volcanic activity decays after some underground activity without any surface activity. This important point has not been solved during this brief experiment, but some pertinent facts are found in our records. In one type of variation of the vertical potential gradient of the earth-current, see Fig. 4, 5 and 6, the variation behaved almost normally before the volcanic activity. But after that, it exhibited immoderate fluctuations. In this case, though the rumble was rather violent, the crater activity ceased without any severe ejection. May this procedure not be taken as a phenomenon of decay of the volcanic activity without a corresponding surface activity (i. e. with underground activity)? If so then as an extreme case, we may expect a special variation type of the vertical potential gradient accompanied by no surface activity. Owing to the dearth of knowledge about this point, we must refrain from making any assertion.

(v) Regarding the influence of climatic factors upon the variation of the vertical potential gradient of the earth-current, the general features of their mode of action and their magnitude have already been clarified in previous reports. Since this influence can be distinguished from the inherent or characteristic variation of the vertical potential gradient of the earth-current, an experienced observer, will not be misted by it.

IV. Discussion and Conclusion

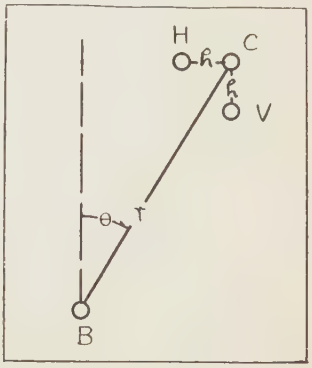
It has been proved that a close correlation holds between the variation of the vertical potential gradient and the volcanic activity of Aso,-that the volcanic activity commences and the pulsatory rumble

takes place when the vertical potential gradient of the earth-current rises and the variation curve reaches its shoulder, and that the activity decays after the variation curve begins to decline. How can this procedure be interpeted?

In this connection, the relation of pressure to the variation of the potential gradient of the earth-current, that " a positive local pressure constitutes a center of negative electric potential," already reported must be remembered. We will apply the fact of this relation to homogeneous earth.

Let us suppose that a positive local pressure has been produced at the center (B) of a sphere of homogeneous earth at constant temperature. B will become the center of a negative electric potential. We will assume that the electric current flows into B at the rate of qp er second to reach a state of equillibrium. The conductivity of the earth is regarded as remaining constant. C and H indicate a pair of horizontal electrodes, and C and V a pair of vertical electrodes. The distance between each pair (h) is taken to be very small as compared with the distance (r) between B and C. If we represent the intensity of the current flowing towards B by i, the total quantity of electricity q will be given by

Fig. 12



$$4\pi r^2 i = q \text{ (constant)}$$

$$i = \frac{q}{4\pi r^2} = K \cdot \frac{d\varphi}{dr}$$

$$\therefore \quad \varphi = -\frac{q}{4\pi k} \cdot \frac{1}{r} + \text{Constant.}$$

Then, the potential difference between C and H and that between C and V will be given by the following equations:

Horizontal Component

$$\varphi_{c-H} = -\frac{g}{4\pi k}\left(\frac{1}{r - h.\sin\theta} - \frac{1}{r}\right)$$

$$= -\frac{q}{4\pi k} \cdot \frac{h}{r} \cdot \frac{\sin\theta}{r - h.\sin\theta} \quad \text{.....(1)}$$

Vertical Component

$$\varphi_{c-V} = -\frac{q}{4\pi k}\left(\frac{1}{r - h.\cos\theta} - \frac{1}{r}\right)$$

$$= -\frac{g}{4\pi k} \cdot \frac{h}{r} \cdot \frac{\cos\theta}{r - h.\cos\theta} \quad \text{.....(2)}$$

Fig. 13

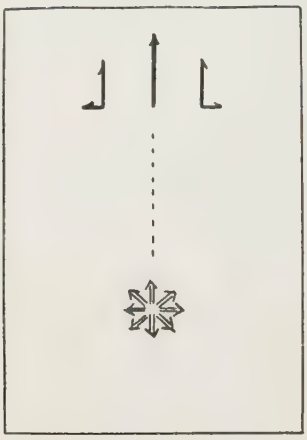
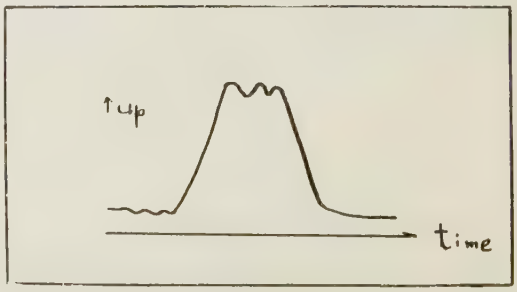


Fig. 13 is a diagram showing a vertical section near the epicenter. The arrows indicate the direction of the rising potential gradient. According to this, in the case of the vertical component, the upper electrode will be higher than the other in potential gradient. and in the case of the horizontal component, the electrode farther from the epicenter will show a higher potential gradient than the nearer one. When translated to our common usage, this means that the vertical component reaches a maximum and the horizontal component a minium at the epicenter.

If so then at the time of an eruption, the storage of explosive substances (mainly gas) should cause high pressure, which occasions the ' up ' change of the vertical potential gradient of the earth-current. As their accumulation in the deep underground requires a long lapse of time, their absolute amount cannot be found from a mere record of the variation of the potential gradient of the earth-current. But when the accumulation reaches a definite amount and the accumulated gas rises and seeks its way out, then the vertical potential gradient will be disturbed and it will fluctuate up and down repeatedly. And when a large amount of gas almost reaches the earth-surface, the distance from the pressure center (r) becomes very small, causing the rapid rise of the vertical potential gradient till it reaches a maximum. After the first eruption, the temporary reduction in pressure brings about the down change of the potential gradient. The supply of gas again makes the electrical potential rise, but once more it will fall after the second eruption. By repeating such processes, a series of explosions will result. But after that, when the supply of gas is exhausted, the vertical potential gradient will keep on falling till it returns to the ininial state and remains there, or for some reason changes to some their point. These procedures are illustrated in Fig. 14.

Fig. 14



A variety of mechanisms concerning the underground activity or an intrusion might be postulated, but it is beyond the scheme of the present paper.

With this reasoning it seems that the data obtained during this term may be interpreted with fair satisfaction. Of course there remain many problems which await closer examination, and the writer hopes for opportunity to go on with the study. The removal of the obsevatory in the crater-wall to a safer place is one of the points to be modified.

In conclusion the writer wishes to express his sincere thanks for the continuous guidance of Prof. T. Nomitu and also for the financial support granted by the Japan Society for the Promotion of Scientific Research. And Mr. M. Hotta, the assistant must be remembered for his enormous efforts in watching the instruments during the study.

The Influence of the Occluded Gases on the Change of Density of Silver by Annealing

By **Atuomi Igata**

(Received, June 10, 1940)

Abstract

By annealing cold rolled pure silver bars successively at higher temperatures, density measurements and microphotographic examination were made at every stage of annealing. It was found with some specimens that by annealing at about 700°—800°C the decrease of density was enormous, and that in this case the occluded gases gushed out of the specimens by opening many small canals along the crystal boundaries during the annealing. The change of the density by the annealing at high temperatures was explained by the assumption that some gases are occluded in small cavities in the specimen, and that these gases expand in the cavities and escape by diffusion through the softened or molten crystal boundaries even when no trace of the small canals was detected.

Silver more than 99.95 % pure was used in this study. It was prepared for the experiment by rolling (to about 50 % reduction in thickness) cast silver bars into long bars, each cross section being 4 mm × 9 mm. Silver specimens cast in vacuum were also used, and these are indicated by the sign (V) in the following figures. When successive annealing for about 3 hours at different temperatures had been done on such a specimen, the density was measured, and the writer found that it generally assumed the typical variation form of A-B-C-D,[1] as is shown in Figs. 1 and 2. Next after subjecting the same specimen to a cold hammering or a cold rolling, the change of its density was measured. Thus have been brought to light[2] somewhat systematic connections between the changes of density and its mechanical treatments.

Now in order to find clearly how the changes of density above mentioned are connected with the internal crystal structure, microscopic photographs were taken of the specimen at various stages of the density curves.

In Fig. 1 the values of density for 4 different specimens (G), (H),

1. A. Igata: These Memoirs, **19**, 215 (1936).
2. A. Igata: These Memoirs, **20**, 35 (1937).

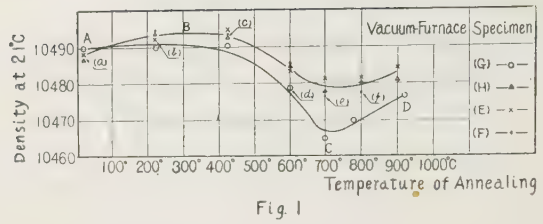


Fig. 1

(E) and (F) are plotted, and the micro-photographs at the stages (a), (b), (c), (d), (e) and (f) are shown in Figs. 1 a, 1b, 1c, 1d, 1e and 1f in Plate I. In this case the decrease of the density at the C state is not very great, and no peculiarity in the microphotographs is observed, except the general growth of the silver crystals with increase of the annealing temperature. Fig. 2 is obtained with two specimens (2) and (3) which were prepared by rolling the silver bars cast in vacuum. In this case the decrease of the density at the C state and its subsequent increase at the D state is very small, and no peculiarity in the microphotographs other than the general growth of the silver crystals with increase of the annealing temperature could be detected as in the case of Fig. 1. The writer has examined many other specimens, and in all cases when the diminution of density at the C state is slight no peculiar difference among the microphotographs at the corresponding stage of the specimens was perceived. But it happened with some specimens that the diminution of density at the C state is very great, as shown in Figs. 3 and 4. By taking microphotographs at the C state in this case it was found that many small holes are formed at the crystal boundaries, through which some gases seem to have gushed out from the inside of the specimen. Figs. 3d and 3e in Plate II are taken at the stages (3d) and (3e) in Fig. 3; and Fig. 4f and 4g in Plate II are taken at the stages (4f) and (4g) in Fig. 4, where the diminution of the density at the C state is enormous, and the holes at the crystal boundaries are much more remarkable.

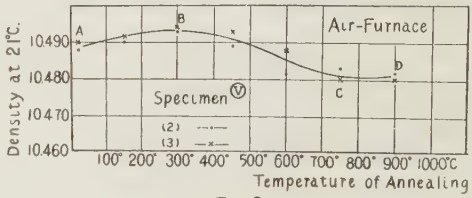


Fig. 2

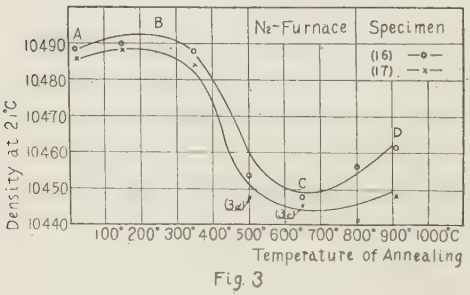


Fig. 3

The furnaces that were used in the present experi-

ments were air-furnace, nitrogen-furnace and vacuum-furnace, and no remarkable difference was detected because of change of the furnace.

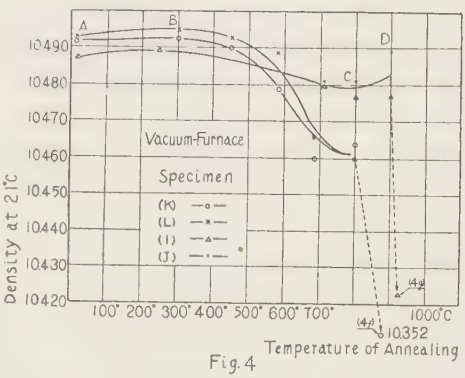


Fig. 4

Silver is said to have a property of absorbing oxygen at a high temperature in the course of refining metallurgy or casting, but it may be presumed that it goes even so far as to contain some other gases besides oxygen in some way of chemical combination, of absorption or of occlusion.

As is described before, violent expulsion of the occluded gases happens only at the crystal boundaries, and not through the crystal itself. This seems to show that a considerable amount of gas is accumulated and confined in narrow spaces of the crystal boundaries. If so, when such a specimen is heated to a temperature as high as 700° or 800°C, the metal will become very soft, especially at the crystal boundaries where it may even melt below the melting point as was pointed out by U. Yoshida and K. Koyanagi[1]; and the occluded gases, having being increased in its pressure to several times of its initial value, will gush out from the inside by opening many canals through the crystal boundaries. Since, judging by the aperture and the number of canals along the crystal boundaries, the amount of the occluded gases is estimated to be enormous, it is thought that a considerable amount of the gases must still be left in the specimen, enclosed in small cavities which are expanded when heated to a high temperature at the C state. The presence of the same circumstances can also be assumed in the case of the specimens with which no trace of the appearance of the small holes along the crystal boundaries at the C state is detected; and thus the decrease of the density at the C state seems to be caused, at least mainly, by the expansion of the small cavities containing occluded gases by being heated to a high temperature at the C state. The amount of decrease of the density depends thus upon the quantity of the occluded gases, and the former increases with the latter.

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1. U. Yoshida and K. Koyanagi: These Memoirs, **18**, 9 (1635).

When the occluded gases are enormous in quantity, they cause a remarkable decrease of the density, and gush out violently by opening small canals along the crystal boundaries. On the contrary when the occluded gas is scanty in quantity, as in the case of the specimen obtained by vacuum casting, the decrease of the density at the C state is very small.

On further annealing the specimen at a temperature higher than that at the C state, the density of the specimen becomes generally greater than that at the C state and this state is denoted by the D state as is shown in Figs. 1, 2 and 3. According to the writer's view, the increase of the density at this state is due to the escape of the occluded gases in small cavities by diffusion through the softened or probably molten crystal boundaries, and to the subsequent partial filling up of the cavities by the softened or the molten metal at the crystal boundaries. As is seen in Fig. 2 and in Figs. 9 and 10 of the former report,[1] the increase of density from the state C to D is small when its decrease at the C state is small. Moreover when the specimen is cold rolled in proper degree after being subjected to the A-B-C-D annealing process, and then annealed again in the same way, the density at the C state in the second annealing process is much greater than that at the first C state and remains almost the same as that at the D state in the first annealing process, as is seen in Fig. 13 of the former report.[2] These facts seem to support the view described above that the recovery of the density at the D state is due to the escape of the occluded gases by diffusion through the crystal boundaries. In the cases of Figs. 3 and 4, where the quantity of occluded gases is supposed to be enormous and some has gushed out through the canals along the crystal boundaries, the gases still enclosed in the cavities will be of considerable amount; and the insufficient recovery of the density at the D state seems to have arisen from this fact together with the circumstance that the formation of many small canals at the C state makes the measured density smaller.

The change of density of cold rolled silver bar at the C and the D state of annealing is thus well explained qualitatively by the occlusion of some gases in the silver specimen. There will still be some other factors involved; and a clear understanding will, of course, be

1. A. Igata: These Memoirs, **19**, 215, (1936).
2. Ibid.

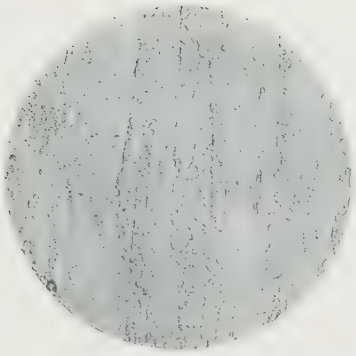
reached after a more exhaustive investigation on this subject; but now it has become evident that an enormous amount of gases is occluded in metals and that the mechanical and physical properties of metals are affected, at least to some extent, by the presence of occluded gases of such amount.

Lastly the writer has estimated the total volume of the small cavities filled with occluded gases by assuming that the decrease of density at the C state is entirely due to the expansion of such small cavities. By taking the true and net density of pure silver as 10.502 at temperature 17°C, and that at the C state as 10.475[1] at 17°C, where no bursting out of the occluded gases has occurred, the total volume of the cavities at the C state becomes about 0.25 % of the volume of silver. In the case of Fig. 4, where the bursting out of the occluded gases has occurred, the density at the C state is about 10.422 at 21°C; and we obtain the value of 0.74 % as the minimum value of the total volume of the cavities. These figures themselves are not remarkable; but when the occluded gases of the volume of this order are dispersed in very tiny cavities rather evenly throughout the metal or along the crystal boundaries, as will actually be the case, their injurious effect on the mechanical properties will be of no inconsiderable degree. As to the pressure of the gases enclosed in small cavities, we must consider it to be very high, because at a temperature as high as at the C state of annealing, the occluded gases burst out of the metal sometimes through the softened crystal boundaries.

In conclusion, the writer humbly expresses to Prof. U. Yoshida of the Kyoto Imperial University his hearty thanks for his kind guidance through all these experiments. Also he should not forget to express his deep gratitude to Prof. Hanemann at der Technische Hochschule in Berlin who has afforded him much assistance in his studies and experiments during his stay in Germany.

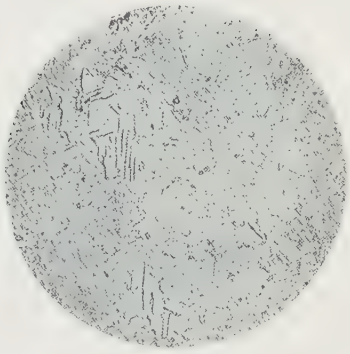
1. A. Igata: These Memoirs, **19**, 215 (1936)

Plate I.



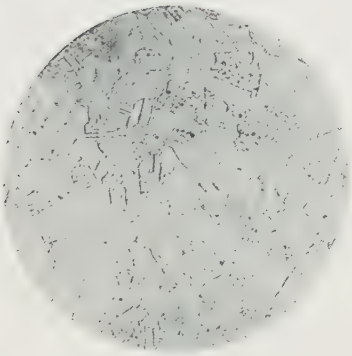
500×

Fig. 1a



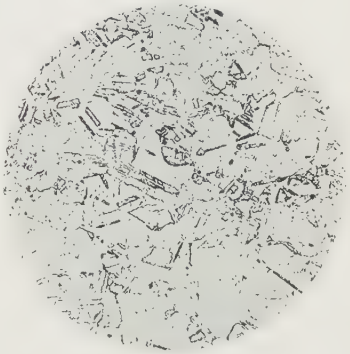
500×

Fig. 1b



500×

Fig. 1c



100×

Fig. 1d

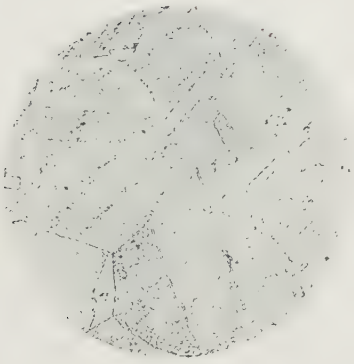


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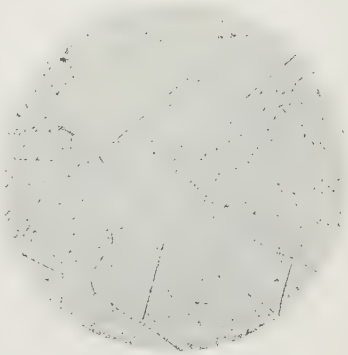
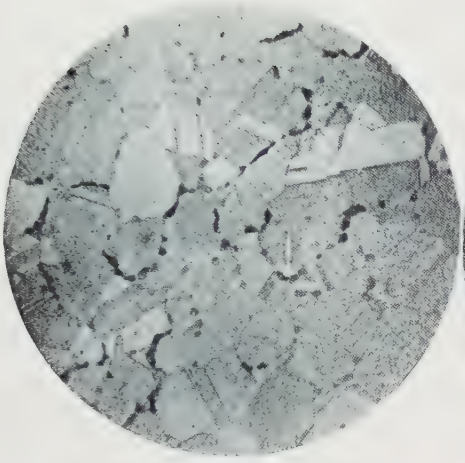
Fig. 1e



100×

Fig. 1f

Plate II.



400×

Fig. 3c

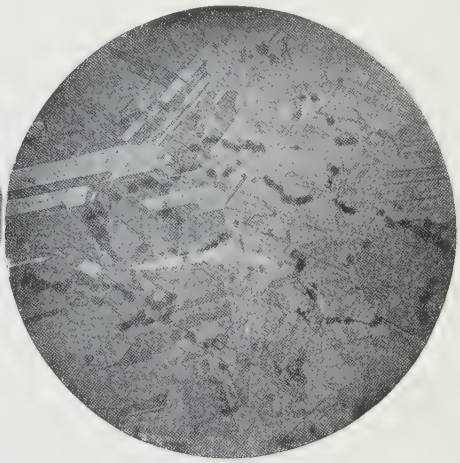


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Fig. 3d

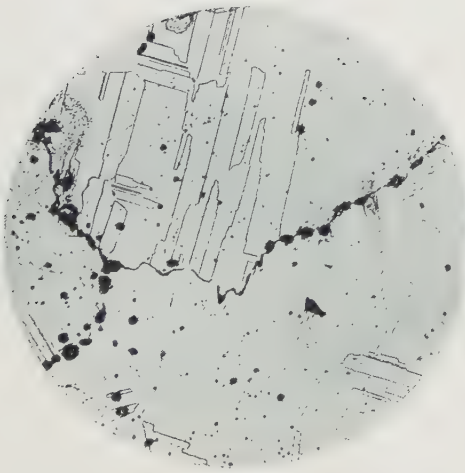

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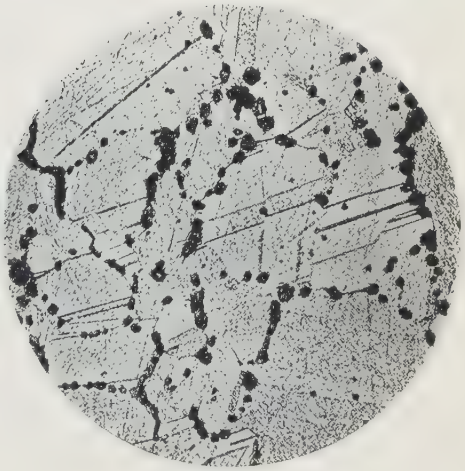
Fig. 4f



100×

Fig. 4g

Molecular Arrangement in Thin Films of Some Paraffins

By Kenzo Tanaka

(Received, June 13, 1940)

Abstract

The arrangement in thin films of molecules of some pure paraffins and paraffin waxes formed on a polished copper plate was examined by the electron diffraction method. By observation of the change of the diffraction pattern with change of temperature, the transition-temperature of the film was measured (i. e. the temperature at which the arrangement of molecules at the surface of the film changes from a regular to an irregular one.) It was found that for pure paraffins the transition-temperature t°C is in linear relation with the number of carbon atoms m in the molecule and is expressed as $t=m+58$. Similar relation was also found for paraffin waxes. The effect of mixing molecules of different chain lengths was observed with paraffin waxes and an empirical expression showing the effect is obtained. The change of the diffraction pattern with temperature is alse discussed.

It is well known[1] that in thin films of some long chain organic substances formed on a flat surface of some solids the molecules are closely packed together side by side with their long axes at a steep angle to the surface of the solid. When these films are heated in vacuum the parallel arrangement of molecules at the surface of the film does not change appreciably even at a temperature somewhat higher than the melting point of the substance, but with further rise of temperature it is gradually destroyed by the heat motion of the molecules and at a certain temperature it becomes haphazard. This course can be followed in the reverse direction by lowering the temperature of the film, and the transition from irregular to regular arrangement of molecules takes place nearly at the same temperature as that of the transition from regular to irregular when the temperature is raised. This temperature has been called the " transition-temperature " of the film[2] and is nearly constant for a given substance.

Experiment

The following are the results of further examination of some pure

1. For instance, C. A. Murison: Phil. Mag., **17**, 201 (1934).
2. K. Tanaka: These Memoirs, **21**, 85 (1938).
„ **22**, 377 (1939).

paraffins (C_mH_{2m+2}) and paraffin waxes by the electron diffraction method. The experimental procedure was the same as before and a small piece of copper plate polished with oooo emery paper was used as the backing of the film in every case. Films of paraffins and paraffiin waxes of different thickness were formed by dropping a drop of benzene solutions of different concentrations of the substance on the surface of the backing and by subsequent evaporation of benzene. Average thickness of the film was estimated from the volume of one drop of the solution and its concentration and the area of the film.

In measuring the transition-temperature by the electron diffraction method, care was taken to avoid the effect of prolonged bombardment of electrons on the same portion of the film since that causes the lowering to some extent of the observed transition-temperature. This was done by changing continuously the portion of the film under examination while it is bombarded or by making the electron bombardment intermittently on the same portion of the film.

Transition-temperature

Measurement of the transition-temperature was carried out with four different films of each substance having the average thickness of 8000 Å. The results are shown in Table I which shows that the transition-temperature is nearly constant for different films of equal thickness of each substance. Among them liquid paraffin did not show any regular arrangement of the molecules in the film treated by the same procedure as others. Parallel arrangement of the molecules in this film was obtained after it was once heated in vacuum at about

Table I

Substance	MeltingPoint °C	Transition-temperature °C			
$C_{17}H_{36}$	22°—22°.5	74°,	74°,	75°,	75°,
$C_{32}H_{83}$	70°—71°	88°,	88°,	88°,	88°
$C_{35}H_{72}$	73°—74°	91°.5,	91°.5,	92°	92°
$C_{64}E_{130}$	102°—102°.5	119°.5,	119°.5,	120°,	120°
Liquid paraffin		60°,	60°.5		
Paraffin wax	36°	70°,	70°,	70°.5,	70°.5
,,	42°—44°	73°.5,	73°.5,	73°.5,	74°
,,	52°—53°	79°,	79°,	80°,	80°
,,	60°—62°	89°,	90°,	89°,	89°
,,	68°—72°	97°,	97°.5,	97°,	97°

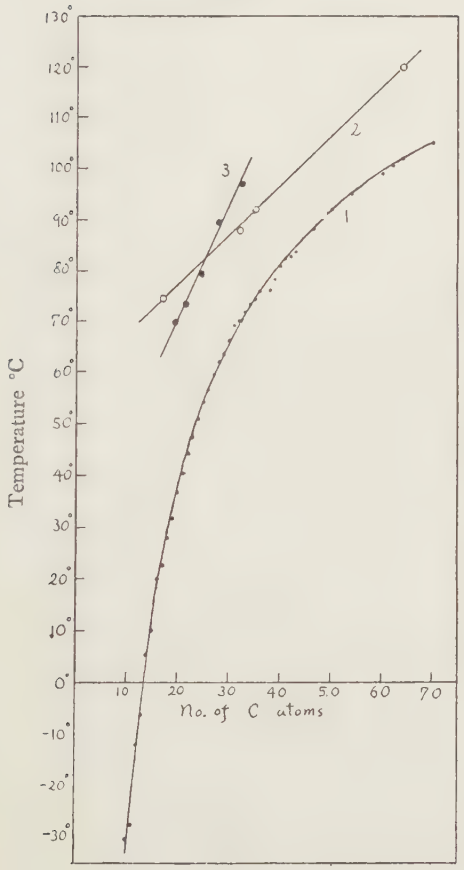
100°C for a while and then cooled below its transition-temperature. Such property was previously observed[1] with some unsaturated fatty acids and some oils.

Table II

Average thickness	Transition-temperature °C			
0.1 mm	72°.5,	72°.5,	73,	73°.5
8000 Å	73°.5,	73°.5,	73°.5,	74°
800 „	74°,	74°,	75°,	75°
80 „	73°.5,	73°.5,	74°,	74°
40 „	73°.5,	73°.5,	74°,	74°

Table II gives the results of measurement with a paraffin wax having the melting range of 42°–44°C. The thickness of the film was changed over a fairly wide range. Since the transition-temperature measured in this case is not markedly different for varying thickness of the film, it may be thought that this temperature is characteristic of the surface layer of the film with the thickness of one or a few molecules, and is one of the important properties characteristic of every one of these substances just as is its melting point.

Fig. 1



In these hydrocarbon molecules it has been shown[2] that the carbon atoms are arranged in a flat zigzag chain and the distance between alternate carbon atoms is 2.54 Å. The length of the molecule is proportional to the number of carbon atoms in the molecule and the increase in length for the addition of one CH_2 group is 1.27Å.

The melting point of the substance belonging to this homologous series increases in a regular fashion with the number of carbon atoms in the molecule as is shown by Curve 1 in Fig. 1. By plotting the transition-temperatures of these

1. K. Tanaka: loc. cit.
1. A. Müller: Proc. Roy. Soc. A, **120**, 437 (1928).

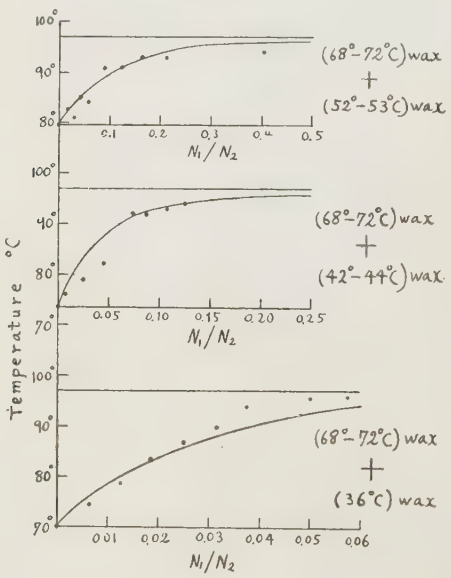
substances shown in Table I in reference to the number of carbon atoms in the molecule, Curves 2 and 3 in Fig. 1 are obtained. Curve 2 is for pure paraffins and Curve 3 is for paraffin waxes. Since paraffin waxes are thoght to be mixtures of molecules having somewhat different lengths the average numbers of carbon atoms in the molecules were estimated from their average melting points. The fact that these two curves are nearly straight lines indicates that the transition-temperature of these substances increases linearly with the number of carbon atoms in the molecule. This fact may be simply explained as follows: As the temperature of the film rises, the mean amplitude of vibration of the molecules increases until kinetic energy of vibration overcomes the attractive force between molecules. At that temperature the transition takes place. Since the attraction acting most effectively between molecules of this kind is a lateral one which may be thought to increase proportionally with the length of the molecule or the number of carbon atoms in it, the transition-temperature should increase linearly with the number of carbon atoms in the molecule.

Curve 2 may be expressed simply as

$$t = m + 58$$

where t is the transition-temperature in °C and m is the number of carbon atoms in the molecule. Addition of one CH_2 group to the molecule causes an increase of 1° in t. Curve 3 has a different inclination with the coordinate axis from that of Curve 2 and this fact seems to be due to the effect of mixing molecules of different chain lengths. The effect of mixing molecules of different lengths upon the transition-temperature can, of course, be observed by mixing pure hydrocarbons of molecules of known lengths in any desired proportions. In the present case this examination could not be performed for lack of pure subs-

Fig. 2



tances. However, the general tendency of the effect may be observed by mixing paraffin waxes of different melting points. For this purpose three different kinds of mixtures of paraffin waxes having different melting points were prepared. These are (68°–72°C) wax + (52°–53°C) wax, (68°–72°C) wax + (42°–44°C) wax and (68°–72°C) wax + (36°C) wax. Transition-temperatures of these mixtures are shown by dots in Fig. 2, where N_1/N_2 in abscissa is the ratio of the numbers of molecules of two kinds of waxes in the mixture. N_1 being for the wax having higher melting point and N_2 for a lower one. The transition-temperature of the mixture approaches very rapidly to that of the higher member with the increase of the number of longer molecules in the mixture. The effect is more marked for larger difference in the lengths of molecules of two waxes mixed, or for larger difference $m_1 - m_2$ in average numbers of carbon atoms in the molecules. The transition-temperature of the mixture may well be expressed in a form

$$t' - t_2 = (t_1 - t_2)(1 - e^{-u})$$

where $u = 0.016\,(m_1 - m_2)^3\, N_1/N_2$ and t_1, t_2 and t' are the transition-temperatures in °C of the two waxes mixed and of the mixture respectively. Curves in Fig. 2 show these empirical relations.

Diffraction Patterns

The diffraction pattern obtained by the reflection of electron beams at the surface of the film does not vary appreciably with the thickness of the film. But when the thickness of the film is reduced to the order of once or twice the molecular length, the whole pattern decreases in intensity and the pattern due to the backing of the film also appears. The diffraction pattern varies also with the temperature of the film. An example of the variation with temperature of the pattern of $C_{32}H_{66}$ is shown by the photographs reproduced in Plate I. Sharp straight layer lines with spots on them as obtained with paraffin molecules at room temperature become diffused and curved and the spots disappear at high temperatures just below the transition-temperature, at which even the diffused and curved lines also disappear. The disappearance of the spots shows the vanishing of a regular packing of the molecules at the surface of the film. The diffused and curved lines seem to be due to the fact that the molecules at the surface of the film are oscillating and are scattered in their orientation independently of each other with small angles about the

direction normal to the surface of the film. At and above the transition-temperature, the mean amplitude of vibration of the molecules becomes so large that the molecules can not arrange themselves in any regular manner, and the diffused curved lines in the diffraction pattern disappear. The manner of variation of the diffraction pattern with temperature is similar in pure paraffins and paraffin waxes. The photograph (f) in Plate I is obtained with a film which was heated above the transition-temperature and then cooled rapidly to the room temperature. The layer lines became diffuse and curved to some extent when compared with those before heating. This seems to be due to somewhat scattered orientation of many small crystals of paraffin, which is caused by a rapid cooling in vacuum from the temperature higher than the transition one to the room temperature. This effect is more pronounced in the case of pure paraffins which are more easily crystallizable than paraffin waxes.

The mean amplitude of oscillation of molecules about a fixed direction at any temperature can be estimated as follows: For simplicity we assume a model of paraffin molecule consisting of m carbon atoms arranged in a straight line with a constant spacing d. The intensity of the electron wave scattered in any direction by such a model is proportional to

$$\psi^2 \frac{\sin^2 \frac{m\delta}{2}}{\sin^2 \frac{\delta}{2}}$$

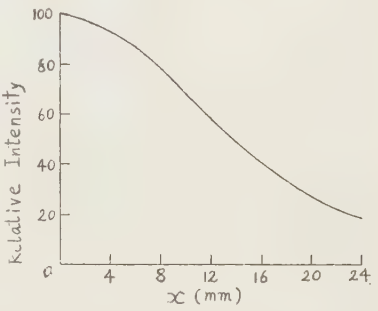
where ψ is the amplitude of the electron wave scattered by a carbon atom and δ is the phase difference between the scattered beams by two successive atoms. The amplitude ψ due to a carbon atom is expressed as

$$\psi = \frac{1}{r} \frac{e^2 m}{2h^2} (Z-f) \frac{\lambda^2}{\sin^2 \frac{\theta}{2}}$$

where Z and f are the atomic number and the scattering factor of the carbon atom respectively, λ the wave length of the electron beam, θ is the scattering angle and others have the usual meanings. The intensity maxima of the scattered radiation appear when $\delta/2$ is an integral multiple of π, at which the value of $\sin^2 \frac{m\delta}{2} / \sin^2 \frac{\delta}{2}$ becomes m^2. The intensity distribution along the second order layer line calculated in the case of $d = 2.54$ Å, $\lambda = 0.05$ Å and D (the distance bet-

ween the specimen and the photographic plate)$=500$ mm, when the incident electron beam proceeds normally to the axis of a molecule standing perpendicular to the surface of the film, is shown in Fig. 3 where x is the distance along the layer line measured from the plane of incidence. The intensity falls fairly rapidly with increasing scattering angle.

Fig. 3



In Fig. 4, using polar coordinate, let the electron beam proceed along the direction OO′ $\left(\frac{\pi}{2}, 0\right)$ upon a molecule whose axis is OP (θ_1, ϕ_1) and let ON be the direction normal to the surface of the film. Next let the diffracted beam OD have a scattering angle θ and make an angle ϕ with the plane of incidence. The diffraction maxima appear when

$$\frac{n\lambda}{d} = \cos\theta_1 \sin\theta \cos\phi + \sin\theta_1 \cos\phi_1 (\cos\theta - 1) + \sin\theta_1 \sin\phi_1 \sin\theta \sin\phi.$$

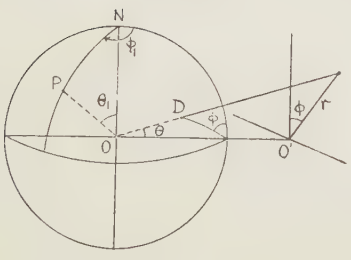
If we represent $\overline{OO'}$ by D and the coordinates of any point on the photographic plate by r, ϕ, the above condition becomes

$$\frac{n\lambda}{d} = \cos\theta_1 \cos\phi \frac{r}{\sqrt{D^2+r^2}} + \sin\theta_1 \cos\phi_1 \left(\frac{D}{\sqrt{D^2+r^2}} - 1\right) + \sin\theta_1 \sin\phi_1 \sin\phi \frac{r}{\sqrt{D^2+r^2}}.$$

As r is very small compared with D in the present case, we may write the above condition as

$$\frac{n\lambda D}{d} = \cos\theta_1 r \cos\phi + \sin\theta_1 \sin\phi_1 r \sin\phi$$

Fig. 4



For given values of n and θ_1 this is an equation of a family of straight lines of parameter ϕ_1 which pass through a common point $(n\lambda D \sec\theta_1/d, 0)$. When all the molecules in the film are at a constant inclination θ_1 and distributed uniformly about the direction normal to the surface of the film, the layer lines are obtained from the above equation by giving various values to ϕ_1 ranging from 0 to 2π. As the incli-

nation θ_1 of the molecule increases the common point recedes far from the direct spot, along the line $\phi=0$ and the angle of divergence of the bundle of layer lines which is equal to $2\theta_1$ becomes larger.

The most intense point on each layer line has the minimum value of θ and is at the foot of the perpendicular drawn on the line from the direct spot. The intensity falls rapidly along the line as the distance from this point increases. For the most intense point on the line obtained with given values of θ_1 and ϕ_1, we get

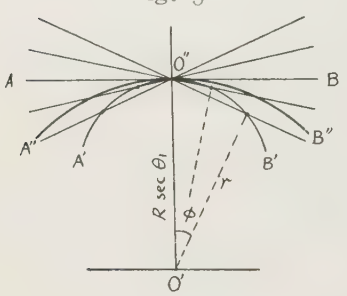
$$r=R\sec\theta_1\cos\phi$$

and

$$\tan\phi=\tan\theta_1\sin\phi_1,$$

where $R=n\lambda D/d$. By changing the value of ϕ_1 for a given value of θ_1 these equations represent a circle having a diameter of $R\sec\theta_1$ as shown by the curve A′O″B′ in Fig. 5. If we assume that the molecules having a constant value of θ_1 are distributed uniformly with respect to ϕ_1, as will naturally be the case, then the layer lines due to them will be a system of an infinite number of straight lines intersecting at the point O″ as is shown in Fig. 5. If the intensity is the same on every point on the lines, the locus of the centre of gravity of the layer lines would be the straight line AO″B. But the intensity at a point on a layer line decreases rapidly as we recede from the foot of the perpendicular drawn from the point O′ to the layer line; and consequently the locus of the centre of gravity of the layer lines will occupy a position between the straight line AO″B and the circle A′O″B′, as is represented by the curve A″O″B″ in Fig. 5. This mode of rough and rather qualitative explanation of the appearance of diffuse and curved layer lines is almost the same as that given by Murison.[1]

Fig. 5



But the actual case will be more complicated. When the temperature is raised to somewhat below the transition-temperature, the inclination θ_1 of the axes of the chain molecules of paraffin will also be scattered to some extent around the ideal orientation $\theta_1=0$. The locus of the centre of the layer lines as stated above will now become diffused, keeping the general aspect of the curve as A″O″B″ unaltered

1. C. A. Murison: loc. cit.

as is actually observed and reproduced in Plate I. In the following the effect of the scattering of molecular orientation will be considered more minutely from the view point of the oscillation of the molecular axes by thermal agitation.

Since any one point on the line $\phi=0$ corresponds to a certain value of θ_1 only, and the intensity of all the layer lines which correspond to various values of ϕ_1 and pass through this point is equal at this point, the intensity at any point on the line $\phi=0$ is proportional to the number of molecules at the inclination θ_1 corresponding to that point. If for instance all the molecules at the surface of the film are in oscillation with a definite angular amplitude Θ, the intensity at a point on the line $\phi=0$ which corresponds to a certain value of θ_1, is inversely proportional to the angular velocity ω of oscillation of the molecule in passing through this inclination θ_1. The intensity is therefore proportional to

$$\frac{\psi^2}{\omega}=\frac{c\psi^2}{(\cos\theta_1-\cos\Theta)^{\frac{1}{2}}}$$

And the intenity distribution is therefore expressed along the line $\phi=0$ by

$$\frac{c\psi^2}{\left(\frac{R}{r}-\frac{R}{r_m}\right)^{\frac{1}{2}}}$$

where $r_m=R\sec\Theta$. This indicates that the intensity is maximum at $r=r_m$ corresponding to the point of the angular amplitude Θ.

The intensity at any one point $(r,\ \phi)$ on the photographic plate depends upon the number of layer lines passing through this point and the intensities of the layer lines at that point. Both extreme lines of the bundle of layer lines passing through the point $(r,\ \phi)$ coincide with the tangential lines drawn from the point to the circle of radius R having the centre at O′. If we denote the intercepts of these two tangential lines with the line $\phi=0$ by r_1 and r_2 respectively, then

$$r_1, r_2=r\cos\phi-r\sin\phi\frac{r^2\sin\phi\cos\phi\pm R\sqrt{r^2-R}}{r^2\sin^2\phi-R^2}.$$

The intensity at the point (r,ϕ) is then proportional to

$$c\psi^2\int_{r_2}^{r_1}\frac{d\rho}{\left(\frac{R}{\rho}-\frac{R}{r_m}\right)^{\frac{1}{2}}}=\frac{c\sqrt{r_m}}{\sqrt{R}}\left[r_m\sin^{-1}\sqrt{\frac{r}{r_m}}\right.$$

$$\left.-\sqrt{r}\ \sqrt{r_m-r}\right]_{r_1}^{r_2}$$

where ρ is the intercept on the line $\phi=0$ of a layer line passing through the point under consideration. The range in which any layer line of the order n exist is given by

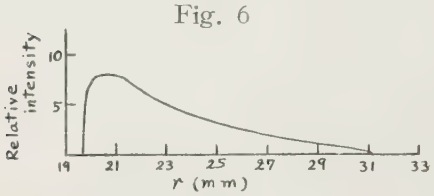
$$\frac{r\cos\phi - r_m}{r\sin\phi} \leqslant \tan\Theta \quad \text{and} \quad r \geqslant R$$

and

$$\frac{r_m - r\cos\phi}{r\sin\phi} \leqslant \tan\Theta \quad \text{when} \quad \phi > \Theta$$

An example of the intensity distribution along the line $\phi=20^\circ$ due to the layer lines of the second order when $\Theta=30^\circ$ calculated by the equations is shown in Fig. 6.

Fig. 6



If we assume that all of the molecules at the surface of the film are oscillating independently of each other with a definite angular amplitude, the amplitude can be obtained by measuging the intensity distribution along any one line of constant ϕ. But the actual case will be more complicated and the amplitude of oscillation will be different for different molecules in the film. Actually the maximum inclination of the molecules about the normal direction of the film does not increase appreciably till the temperature approaches very near to the transition-temperature at which the curved layer lines fade out rapidly. Furthermore in the case of higher member paraffins the spot on the layer lines exists together with the lines up to the transition-temprature where both fade out simultaneously. This shows that a fairly large number of molecules are still arranged in crystal lattice probably due to their stronger lateral adhesion and do not oscillate freely till the transition-temperature is reached. In the following the maximum inclination of molecular oscillation due to thermal agitation was estimated roughly by comparing only the breadth of the curved layer lines in the actual diffraction pattern obtained at any temperature, with that occupied by a system of layer lines which correspond to various values of θ_1 ranging from zero to the maximum value Θ for which also various values taken at the interval of 5° were assigned. The results are shown in Fig. 7. where dots and small circles show the maximum inclinations of the molecules in degrees at those temperatures: representing by a dot the case where the diffraction spots coexist with the layer lines, and by a small circle the case where no spot appears on the layer lines. The relation between

the temperature t and the maximum inclination Θ of molecular oscillation thus estimated was seen to be expressed approximately by

$$\Theta = \frac{2m}{T-t}$$

where T is the transition-temperature and m is the number of carbon atoms in a paraffin molecule. The curves shown in Fig. 7, which were drawn by the above formula agree approximately with actual observation.

The writer wishes to express his sincere thanks to Prof. R. Nozu and Mr. K. Fuchino for their kindness in furnishing pure specimens of paraffin and to Prof. U. Yoshida for his invaluable suggestions given to the writer in carrying out the present experimeat.

Fig. 7

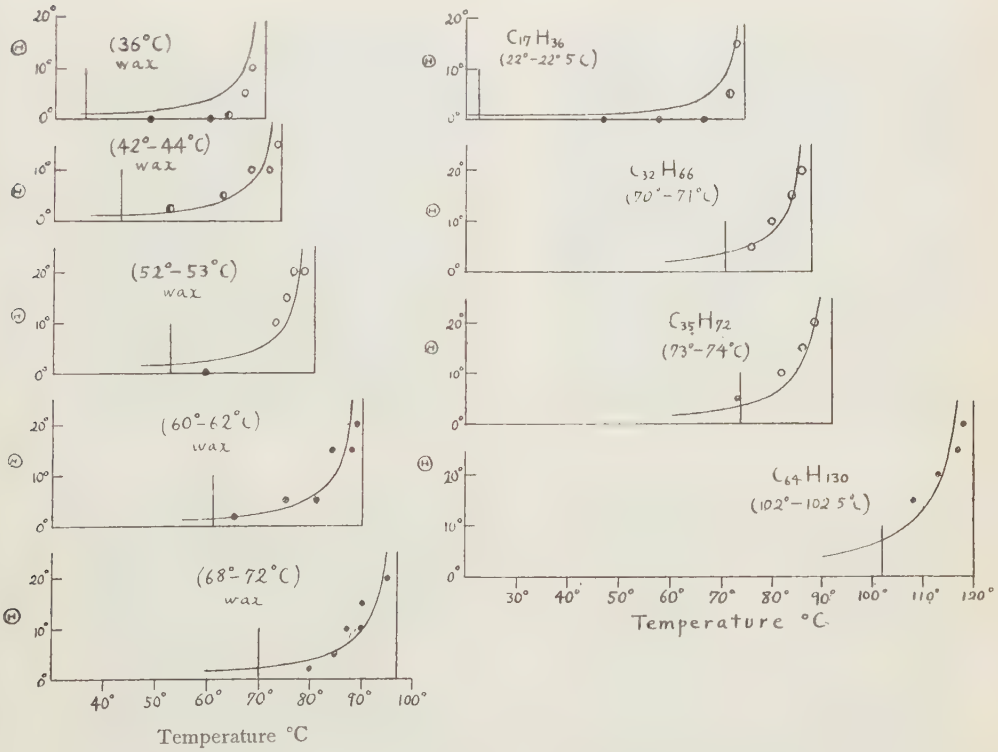
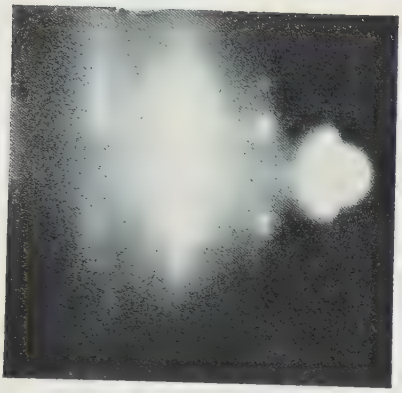
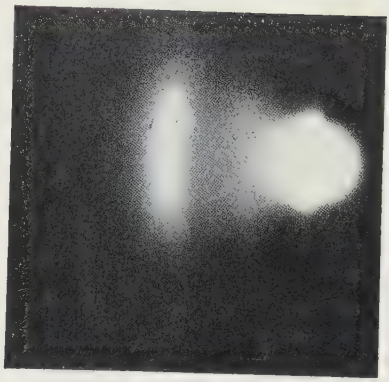


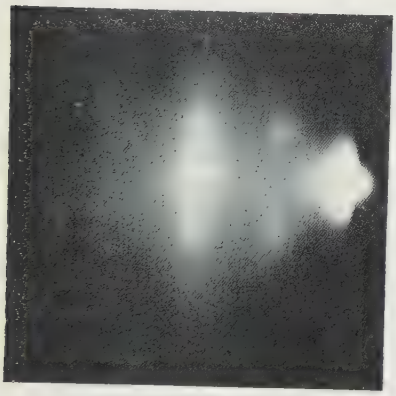
Plate I



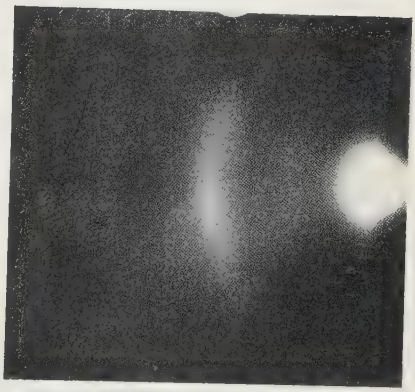
(a) 22.5 C



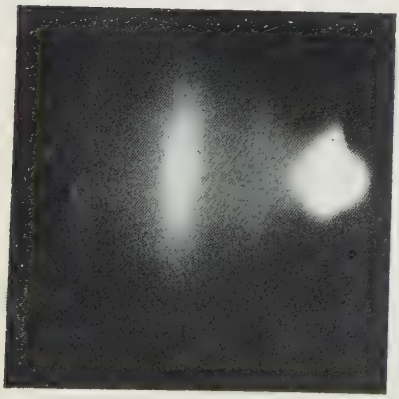
(d) 80°C



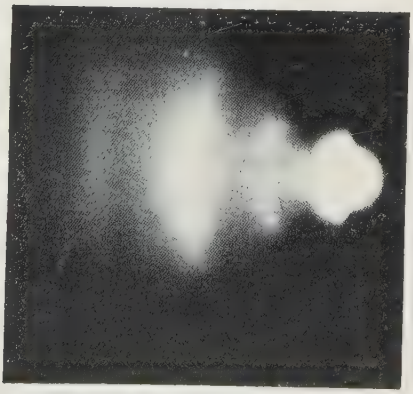
(b) 70.5 C



(e) 86°C



(c) 77°C



(f) 21.5°C

The Phenomena of Crystallization and Melting, and the Internal Structure of Liquid of a Simple Substance

By Usaburo Yoshida

(Received, July 2, 1940)

Abstract

The cybotactic group or the cybotactic crystal in a liquid is thought to be formed by chance in a liquid by sudden grouping of some number of free liquid molecules. Thus the equilibrium between the cybotactic groups and the free liquid molecules can be maintained at any temperature, by changing their relative amounts. This is entirely different from the equilibrium between the adult crystals and their melt, and a simple explanation was looked for. Latent heat of fusion per one constituent molecule is supposed to be much smaller with cybotactic crystals than that of the ordinary adult crystals, and consequently the melting point in the former case is much lower. By considering that the size of a cybotactic crystal is of the order of the unit cell of the crystal, the mechanism of the formation of the crystal nuclei and their growth in a super-cooled liquid is discussed. Anomalous behaviour of liquid water in its specific volume and its viscosity was ascribed to the presence of the cybotactic ice-crystals in liquid water, and a rough amount of the cybotactic ice-crystals was estimated by making use of the temperature dependence of its specific heat and specific volume. To explain the results of the X-ray diffraction with liquid the presence of various kinds of cybotactic groups consisting of different number of constituent molecules was considered. An amorphous solid or the glassy state of a substance was supposed to consist mainly of cybotactic groups, containing some free molecules distributed at random. Beilby layer on polished metal surface was considered to consist of the amorphous solid in the cybotactic sense. Some modification of the theory of recrystallization of metal, proposed previously by the writer and Koyanagi, was made by introducing the idea that the melting point of very fine crystals is much lower, and that at numerous localities in the metal it is transformed to the state of amorphous solid by cold working or by annealing.

Introduction

When a crystal is heated it melts abruptly and completely at its melting point, and we can not heat it above this temperature while keeping it unaltered in the crystallized state. Reversely, when the liquid of the same substance is cooled the circumstances are somewhat different. Usually it crystallizes abruptly and completely at its freezing point which is the same as the melting point of the substance, but in the case of many substances we can, with some care, cool it to a

temperature below its freezing point, producing a super-cooled liquid. The crystallization of the melt without seed crystals takes place only by the growth of the crystal nuclei, which are formed spontaneously here and there in the melt in the super-cooled state. Thus when the nuclei are too scant in number or when their growth is much retarded by some conditions, as when they are too much cooled, the crystallization does not occur; and the melt remains super-cooled for a long time. When we speak of the fusion of crystals, we refer usually to adult crystals which are enormously larger than their nuclei; and the fusion of such crystals takes place on all parts of their boundary surfaces simultaneously, and never at the interior of the crystal.

Cybotactic Group and the Adult Crystal

Some years ago the writer[1] proposed a theory that a great number of tiny ice-crystals having the size of the order of the unit cell of tne ice-crystal are present in liquid water. These tiny ice-crystals are supposed to be unstable and of only temporary existence: a few number of molecules of water unite together to form the crystal lattice only over very short elements of space and time. However the total quantity of such tiny ice-crystals will be determined by the temperature, and it will decrease with rise of the temperature. By taking this fact into consideration. anomalous behaviour of liquid water in its specific volume and viscosity was explained. To clarify the manner of molecular aggregation in liquids the method of X-ray diffraction has been adopted by many workers; and many of them inclined to assume the presence of some kind of pseudo-crystallinity in liquids, a state of temporary arrangement of molecules, very probably similar to that which exists in the solid state. Stewart[2] called such temporary pseudo-crystalline arrangement of molecules in liquid "Cybotaxis". This conception of cybotaxis is similar in general to the writer's idea mentioned above. Though strictly, a distinction must be made between them as will be shown later; but for the present, we regard them as the same in the sense that a spontaneous and unstable aggregation of a certain small number of molecules in a certain manner of configuration occurs momentarily here and there in the liquid in statistical way, and we consider about the equilibrium between the number

1. U. Yoshida: These Memoirs, A, **19**, 271 (1936).
2. Stewart: Rev. Modern Physics, **2**, 116 (1930).

of cybotactic groups and the remaining part of the liquid whose molecules are supposed to be free and not aggregated.

Let the total number of liquid molecules in unit volume be N including both free and aggregated ones, and the number at a certain temperature of the cybotactic groups in unit volume each consisting of x molecules by n; then the total number of free molecules in unit volume will be N-xn. Among these a certain fraction α per unit time will aggregate to form the cybotactic groups, and at the same time a certain fraction β of the cybotactic groups per unit time will decompose into free molecules. Thus, in equilibrium, we have

$$\alpha(N-xn)=\beta xn, \quad \cdots\cdots(1)$$

$$\therefore \quad n=\frac{1}{1+\frac{\beta}{\alpha}}\frac{N}{x} \quad \cdots\cdots(2)$$

In this equation both α and β depend upon the temperature, the number of the constituent molecules x in one cybotactic group and the difference of the potential energy ϕ per one molecule in arranging in the cybotactic group which may depend indirectly upon the number of constituent molecules. However, for a given kind of cybotactic group in a given liquid, the last two factors are constant, and both α and β depend only upon the temperature. With increase of temperature α will decrease and β will increase continuously in consequence of the increase of the kinetic energy of molecules; and thus n will decrease continuously with increasing temperature. This is entirely different from the melting of an adult crystal of a substance into its liquid and the freezing of the liquid on the surface of the adult crystal at its melting point, which take place only at that temperature: i. e. the adult crystal and the liquid of a substance can co-exist only at its melting point, and not at any other temperature. The melting and the freezing in this case take place as has been stated before only at the surface of the adult crystal having a considerable thickness; and thus both the surface density of the molecules constituting the crystal lattice at the boundary surface which are to be dissolved into liquid and that of the liquid molecules which are to be frozen on the surface of the crystal are respectively constant regardless of time and the quantities of the crystal and the liquid. Let N_l be the surface density of the liquid molecules at the surface of the crystal and Nc be that of the molecules of the crystal at its boundary surface,

which correspond to nx in equation (1). Then in equilibrium we obtain, instead of the equation (1),

$$\alpha N_l = \beta N_c, \quad \cdots\cdots(3)$$

which means that the number of molecules which freeze from the liquid to the crystal is the same as that of the molecules which dissolve from the crystal to the liquid, and represents the condition of equilibrium between the crystal and the liquid at the melting point. When temperature is raised even slightly the mean kinetic energy of the constituent molecules in the crystal increases, and the number of the constituent molecules at the crystal boundary whose kinetic energy is greater than is necessary to separate a molecule from the crystal lattice, will become greater. Consequently the value of β increases with temperature. On the other hand, the difficulty of entrapping into the crystal lattice the liquid molecules will become greater with rise of the temperature, due to the increase in their kinetic energy; and consequently the value of α will become smaller with increasing temperature. Thus when the temperature is raised even slightly above the melting point, the right hand side of the equation (3) will become greater than the left, showing that the melting of the crystal proceeds continuously until the entire dissolution of the crystals is reached so long as the temperature is kept higher than the melting point. On the contrary when the temperature is kept even a little below the melting point, the freezing of the liquid goes on until the whole liquid is frozen.

In the case of the cybotaxis the number n of the cybotactic groups per unit volume of the liquid changes continuously with the temperature, and the equilibrium at various temperatures between the cybotactic groups and the free molecules of the liquid is maintained by changing the number of groups as is shown by the equation (2). On the contrary, the surface density of the molecules of the adult crystal and of the liquid at their contact surface, which take part in fusion and freezing, are constant irrespective of the temperature, and the transformation from solid to liquid or vice versa at a temperature other than the melting point can not be balanced to attain an equilibrium as in the case of the cybotaxis. After all, the discontinuous transformation of an adult crystal of a substance to its melt or vice versa at its melting point owes its origin to the fact that it is a surface phenomenon at the crystal boundary, and not at the interior of the crystal,

just as is the case in vaporization and sublimation. The freezing of the liquid of a substance to its crystal, starting from the crystal nuclei, takes place at the very commencement of freezing and sometimes also at a state when the growth of the adult crystals is going on. According to the writer's view the crystal nuclei are nothing but the crystals themselves grown already to a tolerable size as compared to the cybotactic unit cell, though they are still small when considered macroscopically. Thus they are not of spontaneous and unstable existence, and can maintain their crystalline state even at a temperature just below the melting point of the substance. As we may consider their outer surface too in this case, the circumstances are entirely the same as in the case of adult crystals. They have the same power of growth and dissolution respectively at the temperature below and above the melting point as the adult crystals, though the velocity of their growth and dissolution are very much less than those of the adult ones, due to the smallness of their outer surface.

Latent Heat of Fusion and the Melting Point

The latent heat of fusion of the crystal of a substance relates mainly to the work necessary to transfer a constituent molecule of the crystal, which is situated at the crystal lattice on its outer surface, into its melt at its melting point. This work is the same as the difference between the potential energies of the molecule situated in the two different positions. In order to overcome this potential drop, kinetic energy of the constituent molecule due to thermal agitation must exceed it. Thus when this potential drop is great, the melting of the crystal requires a high temperature. This means that the greater the molecular latent heat of fusion of a substance the higher is its melting point. The law that the atomic latent heat of fusion of a substance is proportional to its melting point expressed in absolute temperature is already known to be obeyed roughly by many substances, and is treated theoretically by many authors.[1] This law is obeyed better by the metals,[2] and rather satisfactorily by the metals having the crystal lattices of the face-centered cubic and of the hexagonal closed packed, in which the metals are known to have almost the same relative inter-

1. F. A. Lindemann: phys. ZS., **11**, 609, (1910); E. Grüneisen, Verh. d. D. phys. Ges., **14**, 330, (1912); W. Braunbeck, Z. S. f. Phys. **38**, 549, (1926), etc.
2. I. H. Richards, Chem. News, **75**, 278, (1897).

atomic arrangement. The same law for the boiling point is also valid for many substances and is known by the name of Trouton.

The attractive force of a crystal for an atom or a molecule at its surface to fix the latter into its lattice point will be very much weakened when the size of the crystal is so reduced that it can only exist in the cybotactic state, by reason of the small number of the attracting atoms in the crystal; and the potential drop, against which some amount of work is needed to transfer the atom or the molecule from the crystal lattice into the liquid, becomes smaller. This relation between the potential drop ϕ and the linear size d of the crystal is shown schematically in Fig. 1. The value of ϕ will increase at first with the linear size of the crystal, and will attain a maximum which is the same as that of the adult caystal. If this consideration is correct, the latent heat of fusion of the crystal will be much smaller with very tiny crystals such as cybotactic ones, as compared with that of the adult crystal. According to the law stated above, showing the relation between the latent heat of fusion and the melting point, which must be considered to be strictly valid for the same substance in this case, the decrease of the latent heat of fusion with cybotatic crystals means that their melting point becomes much lower than the ordinary value as was pointed out by Meissner[1] for microscopic crystals. The cybotactic groups are of only spontaneous and statistical existence, and the meaning of their melting point is somewhat different from the ordinary: At the melting point the mean kinetic energy of the molecules due to thermal agitation is the same as the value of ϕ; at a higher temperature the probability or chance for the formation, existence and growth of the cybotactic groups lessens and at a temperature lower than their melting point this probability or chance increases. When a cybotactic crystal in a super-cooled liquid has grown somewhat by chance even at a temperature higher than its original melting point, its melting point rises correspondently, and the chance for its further growth is increased more than bofore. If this happens actually, then its melting point rises again and causes a still further

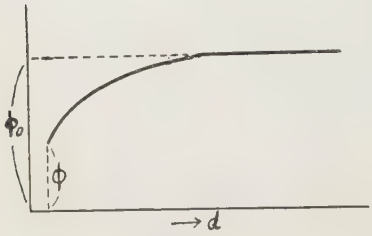


Fig. 1

1. Meissner: Z. S. f. anorg, Chem., **110**, 169, (1920).

increase in chance for further growth. This tendency to the formation and growth of the cybotactic crystals will be greater, of course, than that of becoming thinner or of dissolution into free liquid molecules, at a temperature lower than their melting point, and a rapid crystallization of the whole liquid will occur in this case.

Formation of the Crystal Nuclei and their Growth in a Super-cooled Liquid

If some of the cybotactic crystals in a super-cooled liquid have grown, by chance in the way stated above, to such a size that their melting point is higher than the temperature of the liquid, their stability and the tendency to grow increase more and more one-sidedly, until adult crystals of sufficiently large size are formed in the liquid. Thus the crystal nucleus in a super-cooled liquid is nothing but a submicroscopic crystal which has grown from a cybotactic crystal to such a size that it is already permanently stable and is not in danger at that temperature of being decomposed abruptly by thermal agitation. According to Tammann[1] and others, the number of the crystal nuclei formed spontaneously in a super-cooled liquid attains a maximum at a temperature far below the melting point of the substance, and decreases again with further decrease of the temperature. As this decrease beyond the maximum is ascribed to the enormous increase of viscosity of the liquid at such low temperature, we may consider generally that the chance of the spontaneous formation of the crystal nuclei is more and more favoured with decrease of the temperature, if the effect of increasing viscosity is disregarded. This temperature dependence of the formation of the crystal nuclei is well explained by the idea above stated on the growth of the cybotactic crystals in the super-cooled state. The chance of formation of the cybotactic unit crystals increases with decreasing temperature, i. e. with decreasing kinetic energy of the liquid molecules. An enormous increase of viscosity with decreasing temperature in the super-cooled state thus seems to be due to the increase in number of the cybotactic crystals. If we assume that the growth of the cybotactic crystal is mainly caused by the unification of free molecules of the liquid to the cybotactic crystals existing at that moment, the growth will be retarded at very low temperature by the want of uniting free molecules. This way

1. Tammann: Kristallisieren u. Schmelzen, p. 151, (1903).

of explaining the decrease of the number of the crystal nuclei at very low temperature is likely to be more reasonable than to ascribe it merely to the increased viscosity.

When crystal nuclei have been formed in a super-cooled liquid, they have already acquired a permanent stability in all temperatures below the melting point of the substance, and the change for them is now only toward their growth even up to adult crystals, so long as the temperature is maintained below the melting point. The rapidity of the crystal growth in this state will of course be retarded by the increase of the temperature due the increase of the kinetic energy of the molecules, but at the same time it will be accelerated by the increase of the free molecules colliding on the surface of the crystal with increasing temperature. The fact that the temperature of the maximum rapidity of the crystal growth in the super-cooled liquid is rather shifted toward a temperature higher than that corresponding to the formation of the maximum number of crystal nuclei suggests that the presence of the latter factor influences the crystal growth. If this reasoning be correct, a cybotactic group which is to be developed to a crystal nucleus and even to an adult crystal must have at its birth the same crystal lattice as the adult one. This is somewhat different from the idea of the "pseudo-crystalline" arrangement of the molecules which has been adopted by other authors, and is distinguished by the writer by calling it "cybotactic crystal". According to the writer's view there are many kinds of cybotactic groups, consisting of various numbers of constituent molecules. When the number of constituent molecules is too few, for example two or three, it is insufficient to form a unit cell of the crystal. To form a unit cell of the body-centered cubic, 9 atoms are necessary, and for that of the face-centered cubic 14 atoms are needed. Thus the cybotactic crystal which has the ability to grow to the crystal nucleus must contain at least the number of constituent molecules or atoms of this order.

Cybotactic Crystals in Liquid Water

In the former report[1] the writer has assumed that the cybotactic ice-crystals have the same latent heat of fusion and the same specific volume as those of the adult crystal respectively. But in the present case, the writer has assigned a much smaller value of the latent heat

1. loc. cit.

of fusion to the cybotastic crystals. Water has a peculiar property of expanding by freezing, and consequently that its melting point decreases with increasing pressure. Parallel with these, the specific volume of liquid water has a minimum at 4°C, and increases with decreasing temperature below this point. This temperature of minimum specific volume becomes lower with increasing pressure[1], and the presence of the minimum becomes almost insensible at the pressure of about 2000 Kg/cm². The change of viscosity of liquid water with pressure[2] behaves anomalously also at temperatures below about 30°C. Below about 30°C, the viscosity decreases at first with increasing pressure, attains a minimum, and then increases again with the increase of pressure. If we assume the presence in liquid water, of cybotactic ice-crystals whose specific volume is the same as that of ordinary ice, the specific volume of liquid water will become smaller with increasing temperature, by the diminution of the total number of the cybotactic crystals, and at the same time it will become greater by the thermal expansion with increase of the kinetic energy of the molecules. The predominance of the former effect at low temperature gives rise to the appearance of the minimum in the specific volume at 4°C; and the diminution of number of the cybotactic ice-crystals with increasing pressure shifts the point of minimum specific volume toward the lower temperature. The presence of the cybotactic crystals, which are of course much larger than the free liquid molecules, will increase the viscosity, which will also increase with increasing pressure even when the internal structure of the liquid remains unchanged. Thus the presence of minimum in the viscosity of water at a certain pressure seems to be due also to the diminution of the number of the cybotactic crystals with the increase of pressure. The rate per one degree of temperature change of the diminution in number of the cybotactic crystals with increasing temperature, is considered to be greater at lower temperature, and this will have the effect of making the specific heat of the liquid especially higher at lower temperature than would be expected when no change in the internal structure of liquid occurs with change of the temperature. This is actually observed in the case of liquid water, where the specific heat is minimum at about 35°C. Thus, if we accept the presence of cybotactic ice crys-

1. Bridgeman: The Physics of High Pressure, p. 143, (1931).
2. Hauser: Ann. d. Phys., **5**, 597, (1901); Bridgeman: The Physics of High Pressure, p. 347, (1931).

tals in liquid water, its anomalous behaviour is understood qualitatively.

The next problem is to estimate the quantity of the cybotactic ice-crystals. This is already investigated by the writer[1], and the value of 0.6 % at 0°C was obtained. The general principle in doing this is as follows: The specific volume and the specific heat of liquid water must increase gradually without any abnormality always with increasing temperature, if the effect caused by the presence of the cybotactic ice-crystals in liquid water is eliminated. In regard to the temperature dependence of the quantity of the cybotactic crystals the writer has assumed as before, that the quantity of cybotactic crystals can be represented by an exponential function of the temperature expressed in Centigrade. This is in conformity with Boltzmann's expression for the molecular aggregation, at least for a temperature range not very wide. Thus, for the quantity of the cybotactic crystals x at the temperature t, we have

$$x = x_0 e^{-\alpha t},$$

where x_0 is the quantity of cybotactic crystals at 0°C, and α is a certain constant. The above formula will not be strictly correct for a very wide range of temperature, but it seems reasonable to consider it valid in showing the general tendency of the temperature dependence of the quantity of cybotactic crystals, because the formation and the decomposition of the cybotactic crystals are statistical phenomena.

In the previous report the writer assumed that the latent heat of fusion of the cybotactic ice-crystals is the same as that of the adult ice-crystals and adopted the value of 80 cal. per 1 gr. But this means that the melting point of both the cybotactic and adult crystals is the same, aud is 0°C; and this is inconsistent with the presence of the super-cooled state of liquid water. The logical con-

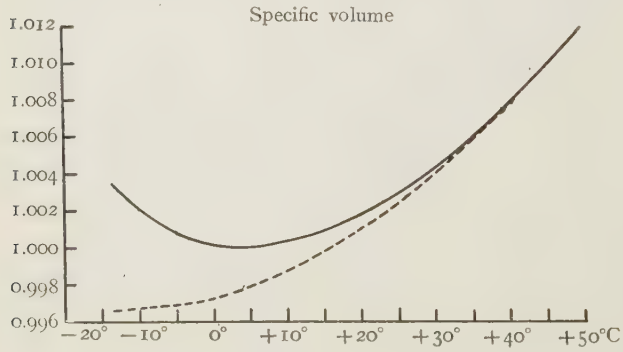


Fig. 2

1. loc. cit.

sequence of the writer's consideration requires that in order for the super-cooled state of a liquid to be possible at a temperature below its freezing point, this temperature must be above the melting point of the cybotactic crystals, and this in turn requires that the latent heat of fusion of the cybotactic crystals is much less than that of the adult crystals. For the latent heat of fusion of the former the value of 16.3 cal. per 1 gr. was adopted in the present case, which corresponds to the melting point of -217.4°C. Corresponding to this value of the latent heat of fusion of the cybotactic ice-crystals, the value of 3 % is taken as the quantity of the cybotactic crystals x_0 at 0°C, which is about 5 times the value assumed in the former report; and the value of 0.063 is taken for α. With these values those of x and $\frac{dx}{dt}$ at any temperature are obtained by the exponential function before stated. The contribution of the cybotactic crystals to the specific volume and the specific heat of liquid water is respectively calculated in the same manner as before by considering the specific volume of the cybotactic and the adult ice-crystals the same; and the results

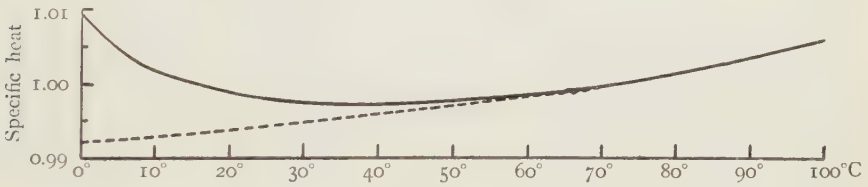


Fig. 3

are shown in Figs. 2 and 3, where the curves drawn in full lines represent the observation and those drawn in broken lines represent the specific volume and the specific heat of idealized liquid water obtained by eliminating the effect of the cybotactic crystals. These curves drawn in broken lines contain no minima, they run gradually without any abnormality always increasing with temperature even in the super-cooled state for the specific volume. This satisfies the conditions required by the idealized liquid water consisting of their free molecules only. At first sight, the above way of estimating the specific heat and the quantity of the cybotactic ice-crystals is entirely arbitrary. But actually this is not so. The writer carried out the same calculation with various values of x_0, α and the latent heat of fusion of the cybotactic crystals taken in a systematic manner, and found that with values other than those given above the conditions required by the idealized liquid water were not fulfilled so satisfactorily. Thus,

though the above estimation of the values of x_0, α and the latent heat of fusion of the cybotactic crystals may not be precise, it may be said to be of a proper order if the writer's view is reasonable.

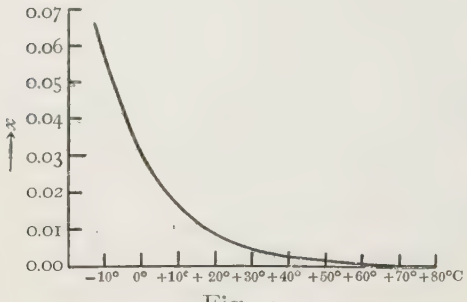


Fig. 4

The temperature dependence of the quantity of the cybotactic crystals in liquid water, which was calculated with the data estimated above, is shown by the curve in Fig. 4. The quantity increases from 0.5 % at about 30°C to 6 % at about −10°C when the temperature is lowered. The viscosity of the liquid water, as is shown by Fig. 5, increases roughly in the same proportion from the value of 0.008 at 30°C to 0.025 at about −10°C. Thus an enormous increase of viscosity with decrease of the temperature, at least in the above temperature range, seems mainly to be due to the increase of the cybotactic ice-crystals.

In the above discussion the writer assumed the presence of only one kind of cybotactic groups, i. e. the cybotactic ice-crystals having the same crystal lattice as the ordinary adult crystal. But it seems more probable that there are several kinds of cybotactic groups consisting of different numbers of molecules. According to Stewart[1] the intensity maxima of the diffraction halo of the monochromatic X-rays due to liquid water can be traced from 2°C to 98° C. He found that the most prominent peak occurs at a gradually increasing angle as the temperature is raised, and that the second peak occurs at smaller angles and finally disappears as the temperature is raised. Bernal and Fowler[2] proposed the theory that there is a tendency for the molecular arrangement at low temperature below 4°C, to be tridimite-ice-like; at ordinary temperatures to be

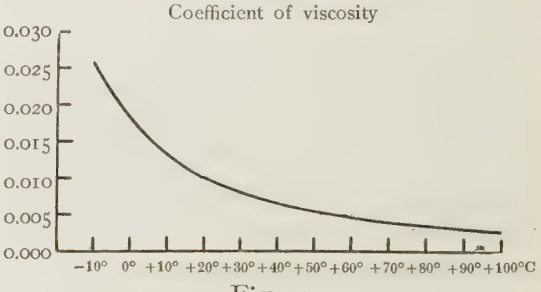


Fig. 5

1. Stewart: Phys. R. **37**, 9 (1931).
2. Bernal and Fowler: J. Chem. Phys., **1**, 515 (1933).

quartz-like; and at high temperature, extending well below the critical temperature of 374°C, there is a tendency to be closed packed, the ideal type of liquid. This idea is the same in its main feature as the writer's. Relative amounts of different kinds of cybotactic groups will change continuously with temperature; and as the diffraction halo is the resultant caused by the mixture of these different kinds of cybotactic groups, the position of its peak will change continuously with temperature, due mostly to the change in their relative proportion. Now let us assume that the work per one constituent molecule which is necessary to decompose a group becomes greater with increasing number of constituent molecules in the group. Then the melting point of a given kind of cybotactic group will be higher with increasing number of constituent molecules in one group; i. e. the stability of the group will increase with increasing number of constituent molecules. On the other hand, the chance of formation of a cybotactic group will increase with decreasing number of constituent molecules. Consequently these circumstances will make the temperature variation of a cybotactic group consisting of a smaller number of constituent molecules much less than that of a cybotactic group consisting of a greater number of constituent molecules. Accordingly the cybotactic groups composed of larger numbers of constituent molecules will more and more prevail in quantity as the temperature is lowered; and on the contrary, those composed of smaller numbers of constituent molecules will become more and more predominant as the temperature is raised. This idea is in accord in general aspect with that proposed by Bernal and Fowler, who have assumed the presence, in liquid water, of three different kinds of cybotactic groups, differing in their relative amounts with different temperatures; and the cybotactic ice-crystals as was assumed by the writer correspond to the tridimite-ice-like groups considered by Bernal and Fowler. According to the writer's estimate, as is stated before, the amount of the cybotactic ice-crystals is small above 0°C, and has already fallen to 1 or 2 % at 10°C. Thus the diffraction halo which was observed by Stewart at temperatures as high as 98°C must be ascribed to the other cybotactic groups containing a smaller number of constituent molecules or to free water molecules distributed at random.

In calculating the specific volume and the specific heat of the idealized liquid water, the writer has paid attention only to the free molecules of water and the cybotactic ice-crystals having a size larger

than the unit cell of the ice-crystals, disregarding the presence of the other cybotactic groups composed of a smaller number of molecules. In these cybotactic groups composed of fewer molecules, the constituent molecules are thought to be united much more loosely than in the cybotactic ice-crystals; which is equivalent to saying that the latent heat of fusion per one molecule is much less in the former than in the latter. Moreover, it may be supposed that, when the intermolecular cohesion is small in a cybotactic group and does not differ much from that among free water molecules, the average distance between the neighbouring molecules and consequently the specific volume of water will be nearly the same in the two cases. Consequently, the specific volume and the specific heat of the idealized liquid water will be those of the mixture composed of the free molecules and the cybotactic groups of weaker intermolecular cohesion, obtained by eliminating the effect of the cybotactic ice-crystals as was stated before.

Glassy State or Amorphous Solid

It was commonly considered that the glasses are amorphous or super-cooled liquids. Recent investigation on amorphous solids of many substances[1] with X-ray diffraction, has revealed the fact of an orderly arrangement of constituent atoms or molecules similar to that observed in the crystal of the same substance. The diffraction halo due to an amorphous solid is diffuse and broad. In this respect the amorphous solid resembles the liquid. On the other hand the position of the intensity maxima of the diffraction halo of an amorphous solid, coincides with that of the centre of average intensity of sharp and prominent diffraction lines due to the assemblage of an immense number of powdered crystals of the same substance, suggesting that the so called amorphous solid is composed of very small, submicroscopic crystals of the same substance. This suggestion seems to be half correct, but not entirely; because a considerable difference in density between the amorphous solid state and the crystalline state of the same substance is detected in many cases, being mostly smaller in the former state. Ordinarily, the word "amorphous" implies a random distribution of the constituent molecules. Thus the two ideas regarding the glassy state as amorphous and as crystalline are in themselves

1. See for example; Randall: The Diffraction of X-rays and Electrons by Amorphous Solids, Liquid and Gases (1934).

self-contradictory in the ordinary sense. However if we extend the writer's theory of the internal constitution of a super-cooled liquid to the temperature range very far below the freezing point of the substance, this self contradiction is completely reconciled. As is stated before, a super-cooled liquid of a substance is composed of two constituents: the free molecules and the cybotactic crystals. As to the relative amounts of these two constituents, the latter increases with decreasing temperature. When the temperature is so decreased as to be lower than the freezing point of the cybotactic crystals, the chance or probability of their spontaneous formation and of their growth becomes greater than the chance or probability of their decomposition into free molecules. Thus in this state, the major part of the constituent molecules will be united in the cybotactic crystals, leaving a minor part as free molecules; as is schematically represented by Fig. 6. The stability or the probability for persistence of an individual cybotactic crystal is of course very good in this case; but it is not absolute, and on the average a small degree of transformation from the cybotactic state to that of free molecules or vice versa is to be considered to be going on from time to time. However, the relative proportion of the total amounts of the cybotactic crystals and the free molecules is a statistical problem, and is determined by the temperature. This is very different from the case of adult crystals, and is the reason that the amorphous solid has no sharp melting point. If this reasoning is correct, the amorphous solid or the solid in glassy state is simply a super-cooled liquid at a temperature, usually the room temperature, which is lower than the melting point of the cybotactic crystals. The resemblance of the x-ray diffraction halo caused by an amorphous solid of a substance to that caused by the powdered crystals of the same substance is now explained, if we consider that the halo is mainly due to the cybotactic crystals, which are the main composition of the amorphous solid and are supposed to consist mostly of cybotactic crystals having a size larger than the unit cell of the crystal of the substance at such a temperature. The cybotactic crystals or the groups are formed spontaneously here and there in the liquid. Thus when the larger part of the liquid is occupied by the cybotactic crystals, the remaining molecules, which can not occupy the lattice points, will be left and confined irregularly among the narrow and irregular interstices between neighbouring cybotactic crystals. This

Fig. 6

seems to explain the remarkable difference between the densities of the amorphous solid of a substance and its crystal.

If the internal structure of an amorphous solid is like that sketched above at room temperature, then the melting point of the constituent cybotactic crystals will be much higher than the room temperature, and the growth of some of the cybotactic crystals may of course be possible by taking in neighbouring free molecules. But the cybotactic crystals are too numerous and consequently too close together, and the number of the free molecules situated at the crystal surfaces is too scant for such growth actually to occur; and the crystal growth i. e. the devitrification of glassy state happens at higher temperature. This is entirely the same as in the case of a super- cooled liquid which is not solidified.

When an amorphous solid is not completely solidified and is somewhat soft at a certain temperature, we can easily deform it plastically by applying a proper force slowly and continuously, i. e. the solid has some fluidity. But at the same time, it is brittle or fragile for a sudden shock, and we can break or shatter it into pieces by applying a sudden impact of sufficient strength. In such a state of the amorphous solid the larger part of the body will be in the state of the cybotactic crystals, but the transformation from the cybotactic to the free molecules and vice versa will be going on in some degree throughout the whole portion of the solid, and mutual displacements between all neighbouring small portions of the solid may take place slowly if compelled under the application of a sufficiently gradual force, by virtue of the cybotactic transformation. When the impact applied is too sudden and strong, the solid will not have enough time to accomplish the necessary mutual displacements between neighbouring portions in it, by the mutual transformation between the two kinds of molecular aggregation, i. e. the cybotactic crystals and the free molecules; and thus it is fragile under the sudden impact.

Beilby Layer on Polished Metal Surface

Beilby's hypothesis that the well polished surface of a metal consists of an amorphous layer has recently been tested by many authors by means of the electron diffraction. From the presence of the diffuse and broad halo, characteristic of the liquid, some have thought that they confirmed the correctness of the Beilby hypothesis. On the other hand, it may be imagined, from the approximate coincidence of

the position of the intensity maxima of the diffraction halo with that of the prominent diffraction lines due to powdered crystals of the metal, that the polished surface layer consists of very small metal crystals having a size not very much larger than that of the unit cell of the crystal. This point is not yet clarified, but it seems to be solvable by applying to this case the writer's idea of the internal structure of the solidified state of a super-cooled liquid. The main difficulty in doing this is that the melting of the surface layer by the process of abrasion and polishing should occur before solidification. But this will be easily overcome if we consider that the metal crystals at the surface are distorted and are smashed into very fine crystals having the size of the order of the unit cell of the crystal by the abrasion and the subsequent fine polishing, and consequently their melting point decreases enormously as was estimated in the case of the cybotactic ice-crystals. If so, local heating by subsequent polishing to a temperature above the so-decreased melting point may easily melt the fine crystals; and the solidified state of the super-cooled liquid of the metal may be produced by very rapid cooling occurring actually at the same time as the local heating. The fact that the electron diffraction halo obtained with a very finely polished metal surface indicates the presence of a half amorphous and half crystalline state, seems thus clearly explained, in accord with the writer's theory, by considering the formation of a solidified state of super-cooled liquid at the surface of the metal during the prosess of fine polishing.

The diffraction halo observed with the amorphous solid or the Beilby layer of a substance corresponds, in main features, to the diffraction lines obtained with the powdered crystals of the same substance. But this correspondence is not exact. With a cybotactic crystal having the size of the order of the unit cell of the crystal, the atomic density in an atomic plane which reflects the x-rays or the electron wave is entirely different from that in the same atomic plane of an adult crystal, even in the case of the powdered ones which are enormously larger than a cybotactic crystal. As we are ignorant of the actual form of the cybotactic crystals, we can not enter into further details; but, at least, we must expect that some difference in relative intensities of the diffracted waves caused by various atomic planes will take place in both cases, i. e. with cybotactic crystals and with powdered crystals.

Recrystallization of Cold Worked Metals

Lastly the writer will give some modification of the theory of recrystallization of a cold worked metal, which was proposed by the writer and Koyanagi[1] several years ago. On that occasion it was assumed that the metal crystals melt more easily at their crystal boundaries, and when the lattice distortion is greater; and that the crystal nuclei, which are to grow to adult crystals by annealing, are formed spontaneously in the part thus melted by annealing at the crystal boundaries. To these assumptions the writer will add one more: i. e. the finer the crystals the lower their melting point as was assumed before. By cold working the metal crystals at different and numerous localities in the metal will be smashed into very fine crystals, even to the size of the order of the cybotactic crystals; and at the same time most of the fine crystals formed by smashing will be distorted to some extent. By being severely heated at various localities in the metal by subsequent cold working some of the very fine and distorted crystals at these localities will even be melted and cooled very rapidly by virtue of their remarkable low melting point, so that these portions come to the solidified state of the super-cooled liquid, whose internal constitution will be such as was pictured before by the writer. These localities in the solidified state of the super-cooled liquid have no definite melting point, and may also be said to be in the state of amorphous solid or in the glassy state in the sense the writer has previously indicated. When the metal in this state is annealed at a proper temperature these localities in the state of amorphous solid will be softened, and at the same time the other localities consisting of very fine crystals will be melted; and some crystal nuclei will be formed there. By continued annealing these nuclei will grow gradually in the softened part, and at the same time these softened amorphous parts will proceed further and further by consuming the remaining distorted fine crystals or the other softened amorphous parts. This is an outline of the writer's idea of the recrystallization of cold worked metals.

The writer is grateful to the special fund of the Department of Education for the financial support of the present work.

1) U. Yoshida and K. Koyanagi: These Memoirs, A, **18**, 9 (1935)

Sur l'Unicité des Solutions d'un Système d'Équations différentielles ordinaires

Par **Hirosi Okamura**

(Reçu le 8 Juillet 1940)

Nous allons exposer une théorie nouvelle sur l'allure des solutions d'un système d'équations différentielles ordinaires et nous appliquerons cela à démontrer une condition nécessaire et suffisante pour l'unicité d'une courbe solution issue d'un point donné, condition analogue à celle que nous avons trouvée autrefois à propos d'une seule équation[1] et que nous n'avons pas pu généraliser depuis à un système d'équations. La méthode de raisonnements est extrêmement plus directe qu'au Mémoire précédent, tandis que pour le cas particulier d'une seule équation la condition actuelle est moins forte que l'autre, ce que nous signalerons finalement.

I. Cas d'un système d'équations différentielles définies par une famille de courbes.

Commençons par un cas simple mais important où se manifeste visiblement notre idée fondamentale.

Soit Ω une famille de courbes, telles que chacune d'elles est définie par un système d'équations de la forme

$$y_i = y_i(x) \quad (i = 1, 2, \ldots, n),$$

où $y_i(x)$ sont des fonctions continues réelles de x, ainsi que leurs dérivées premières, pour $0 \leqq x \leqq 1$, et que toutes ces courbes sont situées dans un ensemble de points $(x, y_1, \ldots, y_n)$ borné et fermé, E, de façon que par chaque point de E il passe une seule courbe de la famille Ω. Les directions de ces courbes définissent un système d'équations différentielles

$$\frac{dy_i}{dx} = \omega_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n)$$

dans E. *Nous supposerons de plus* que ces fonctions ω_i $(i = 1, 2, \ldots, n)$ sont continues dans E.

Concernant ces équations différentielles nous étudierons la distribution des points de Peano, ce qui revient à chercher des courbes qui sont en tous leurs points tangentes aux courbes de la famille Ω

1. Okamura, ces Memoirs, A **17** (1934), p. 319 [Correction, A **19** (1936), p. 269].

mais étrangères à Ω.[1] Pour cela. il nous faut une notion fondamentale jouissant d'un rôle principal, que nous exposerons dans la suite, avant d'énoncer le résultat.

Convenons d'abord de désigner par $C(P)$ la courbe de la famille Ω qui passe par le point P.

Soient P et Q deux points quelconques de E, P étant sur l'hyperplan $x=\xi$, Q sur $x=\xi'$, où nous supposerons que $\xi \leqq \xi'$. Divisons le segment $[\xi, \xi']$ en ν parties par des points ξ_k tels que

$$\xi=\xi_0 \leqq \xi_1 \leqq \ldots \leqq \xi_\nu=\xi'.$$

Prenons un point Q_k de E tel que $x=\xi_k$, et désignons par P_{k+1} le point où la courbe $C(Q_k)$ coupe l'hyperplan $x=\xi_{k+1}(k=0, 1, \ldots, \nu-1)$. Posons

$$S=\overline{P_0Q_0}+\overline{P_1Q_1}+\ldots+\overline{P_\nu Q_\nu} \quad (P_0=P,\ Q_\nu=Q)$$

où $\overline{P_kQ_k}$ est la longuer euclidienne du segment P_kQ_k. Pour deux points donnés P et Q, on peut choisir d'une infinité de manières le nombre de division ν, les points de division $\xi_1, \ldots, \xi_{\nu-1}$ et les points Q_k de E tels que $x=\xi_k$. Nous désignerons par $D(P, Q)$ la borne inférieure de telle somme S possible et nous poserons, par définition, $D(P, Q)=D(Q, P)$.[2]

La fonction $D(P, Q)$ étant introduite, nous avons le théorème suivant qui achève notre recherche pour le cas actuel :

Pour que deux points P et Q de E se trouvent sur une même courbe solution du système des équations différentielles, il faut et il suffit que l'on ait

$$D(P, Q)=0.$$

En effet, cette condition est nécessaire parce que, si une courbe solution des équations différentielles, C, passe par deux points P et Q pour lesquels $x=\xi$ et ξ' avec $\xi \leqq \xi'$, alors, en formant la somme S au moyen des points Q_k pris sur C tels que $x=\xi+\frac{k}{\nu}(\xi'-\xi)$ $(k=0, 1, \ldots, \nu-1)$, la somme S tend vers 0 avec $\frac{1}{\nu}$, en vertu de la continuité des ω_i, et par suite $D(P, Q)=0$.

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1. La possibilité de l'existence d'une telle courbe se montre par l'exemple simple $(n=1)$: la droite $y=0$ pour la famille des courbes $y=(x+c)^3$ dépendant d'un paramètre c, qui définissent l'équation différentielle $\frac{dy}{dx}=3y^{\frac{2}{3}}$.

En outre, nos raisonnements subsisteront de même si la dimension de E est inférieure à $n+1$; c'est, par exemple, une telle circonstançe qui arrive, au cas $n=2$, avec la famille de générateurs (Ω) d'une surface développable (E), dont l'arête de rebroussement est le lieu des points de Peano.

2. Lorsque $\xi=\xi'$, $D(P, Q)$ signifie la distance des deux points P et Q.

Réciproquement, supposons que $D(P, Q)=0$ où P est un point de E sur l'hyperplan $x=\xi$, Q sur $x=\xi'$, et $\xi<\xi'$.[1] Alors on a une suite de valeurs de S, $S^{(\mu)}(\mu=1, 2, \ldots)$, tendant vers 0 avec $\frac{1}{\mu}$, où $S^{(\mu)}=\sum_{k=0}^{\nu_\mu} P_k^{(\mu)}Q_k^{(\mu)}$, les x des points $P_k^{(\mu)}$ et $Q_k^{(\mu)}$ étant $\xi_k^{(\mu)}(\xi=\xi_0^{(\mu)}\leqq\xi_1^{(\mu)}\leqq\ldots\leqq\xi_{\nu_\mu}^{(\mu)}=\xi')$. Définissons les fonctions

$$y_i=Y_i^{(\mu)}(x) \quad (i=1, \ldots, n)$$

pour $\xi\leqq x\leqq\xi'$, telles que ce système représente la coube $C(Q_k^{(\mu)})$ pour $\xi_k^{(\mu)}\leqq x<\xi_{k+1}^{(\mu)}(k=0, 1, \ldots, \nu_\mu-1)$ $(x>\xi)$ et qu'il donne les points P et Q pour $x=\xi$ et ξ'. Ces fonctions sont discontinues pour $x=\xi_k^{(\mu)}$ au plus et nous désignerons par $\sigma_i^{(\mu)}(x)$ la somme des discontinuités de $Y_i^{(\mu)}(x)$ pour $[\xi, x]$ de sorte que les différences $Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)$ sont continues pour $\xi\leqq x\leqq\xi'$ et que nous avons évidemment

$$|\sigma_i^{(\mu)}(x)|\leqq S^{(\mu)}.$$

On a d'ailleurs, pour $\xi\leqq x\leqq\xi'$,

$$Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)=Y_i^{(\mu)}(\xi)-\sigma_i^{(\mu)}(\xi)+\int_\xi^x \omega_i[t, Y_1^{(\mu)}(t), \ldots, Y_n^{(\mu)}(t)]dt.$$

Par conséquent la suite des fonctions $Y_i^{(\mu)}(x)-\sigma_i^{(\mu)}(x)$ est également continue, donc d'après le théorème d'Ascoli et Arzèla on peut en extraire une suite uniformément convergente, et l'on a, comme limites des égalités précédentes,

$$Y_i(x)=Y_i(\xi)+\int_\xi^x \omega_i[t, Y_1(t), \ldots, Y_n(t)]dt\,;$$

car $\sigma_i^{(\mu)}(x)$ tendent vers 0 uniformément. Ainsi nous avons une courbe solution $y_i=Y_i(x)$ des équations différentielles passant par les points P et Q. *C. q. f. d.*

De ce théorème il s'ensuit que *pour que le système des équations différentielles $y'_i=\omega_i$ n'ait pas de points de Peano, il faut et il suffit que l'équation $D(P, Q)=0$ entraîne que les points P et Q se trouvent sur une même courbe de Ω.*

D'après cette proposition on prouvera facilement le fait suivant: *Si la famille de courbes Ω est donnée par les équations*

$$\Phi_i(x, y_1, \ldots, y_n)=C_i \quad (i=1, \ldots, n)$$

où C_i sont paramètres et où les fonctions Φ_i vérifient la condition de Lipschitz par rapport à $(y_1, \ldots, y_n)$, alors il n'existe pas de points de Peano et la famille Ω épuise toutes les solutions de ses équations différentielles $y'_i=\omega_i$.

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1. Le cas $\xi=\xi'$ est trivial puisqu'alors $D(P, Q)=0$ exige que le point P coïncide à Q.

Le résultat donné par M. van Kampen[1] relatif à un système dynamique est un cas particulier de cet énoncé.

II. Cas des équations différentielles données à priori.

Considérons maintenant un système d'équations différentielles

$$\frac{dy_i}{dx}=f_i(x, y_1, \ldots, y_n) \quad (i=1, \ldots, n),$$

où nous supposerons que *les fonctions f_i sont continues dans un ensemble de points $(x, y_1, \ldots, y_n)$ borné et fermé, F, tel que la projection de F sur l'axe des x est un segment.*[2]

Soient P et Q deux points quelconques de F, tels que $x=\xi$ et ξ' et que $\xi \leqq \xi'$. Divisons le segment $[\xi, \xi']$ en ν parties de manière que

$$\xi=\xi_0 \leqq \xi_1 \leqq \ldots \leqq \xi_\nu=\xi'.$$

Prenons un point Q_k de F sur l'hyperplan $x=\xi_k$ et menons une droite passant par Q_k et ayant les coefficients angulaires donnés par les valeurs de f_i au point Q_k, qui coupe l'hyperplan $x=\xi_{k+1}$ en un point P_{k+1} $(k=0, 1, \ldots, \nu-1)$. Posons

$$\Delta=\overline{P_0Q_0}+\overline{P_1Q_1}+\ldots+\overline{P_\nu Q_\nu} \quad (P_0=P,\ Q_\nu=Q).$$

Pour deux points donnés P et Q, considérons toutes les valeurs possibles d'une telle somme Δ et prenons la plus petite des limites de Δ, en faisant tendre vers 0 la valeur maximum des différences $\xi_k-\xi_{k-1}(k=1, \ldots, \nu)$.[3] Cette limite est, au cas particulier où $f_i=\omega_i$ et $F=E$, identique à $D(P, Q)$ précédemment définie, ce qu'on peut voir facilement. Ainsi nous avons généralisé la notion $D(P, Q)$ et nous conserverons la même notation $D(P, Q)$ pour le cas actuel. Posons encore, par définition, $D(P, Q)=D(Q, P)$.

Alors, comme au cas précédent, par un raisonnement légèrement plus compliqué, seulement en remplacant des segments Q_kP_{k+1} au lieu

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1. Amer. J. Math. **59** (1937), p. 144.

2. On peut supposer qu'à plus forte raison, l'ensemble F soit d'un seul tenant.

3. Pour obtenir effectivement cette limite, on procédera comme il suit: On forme Δ pour ν fixe, compatible avec la condition

$$\xi_k-\xi_{k-1} \leqq \epsilon \quad (k=0, 1, \ldots, \nu-1),$$

ϵ étant un nombre positif; alors Δ est une fonction continue de $(\xi_1, \ldots, \xi_{\nu-1}, Q_0, \ldots, Q_{\nu-1})$ et atteint son minimum $\Delta_\nu^{(\epsilon)}$. Ce minimum décroît en même temps que $\frac{1}{\nu}$ et l'on pose

$$\Delta^{(\epsilon)}=\lim_{\nu\to\infty}\Delta_\nu^{(\epsilon)}.$$

Cette limite $\Delta^{(\epsilon)}$ augmente à son tour si l'on fait tendre ϵ vers 0 et elle tend vers la limite que nous avons cherchée.

des arcs $C(Q_k)$, nous avons le théorème suivant, plus général qu'au paragraphe précédent :

Pour que les deux points P et Q de F se trouvent sur une même courbe solution des équations différentielles $y_i'=f_i$, il faut et il suffit que l'on ait

$$D(P, Q)=0.$$

De plus, cette fonction $D(P, Q)$ possède, d'après sa définition, quelques propriétés remarquables, qui nous seront utiles dans le paragraphe suivant. Par exemple, on a les inégalités, faciles à vérifier, pour trois points de F, P, Q et R de coordonnées $(\xi, \eta_1, \ldots, \eta_n)$, $(\xi', \eta_1', \ldots, \eta'_n)$ et $(\xi'', \eta_1'', \ldots, \eta_n'')$ où $\xi\leqq\xi'\leqq\xi''$,

$$D(P, R)\leqq D(P, Q)+D(Q, R),$$

$$D(P, Q)\leqq D(P, R)+M(\xi''-\xi')+\sqrt{(\eta_1''-\eta_1')^2+\ldots+(\eta_n''-\eta_n')^2},$$

$$D(Q, R)\leqq D(P, R)+M(\xi'-\xi)+\sqrt{(\eta_1'-\eta_1)^2+\ldots+(\eta_n'-\eta_n)^2},$$

M étant la borne supérieure de $\sqrt{f_1^2+f_2^2+\ldots+f_n^2}$ dans F. Il en résulte la continuité de la fonction $D(P, Q)$ par rapport à deux points P et Q intérieurs de F, ou sous certaines réserves, à la frontière de F. Il est à remarquer, cependant, que $D(P, Q)$ n'est pas toujours finie à moins que F ne soit assez régulier.

III. Théorème d'unicité.

Nous arrivons maintenant à la dernière étape du Mémoire pour démontrer le théorème d'unicité suivant :

Supposons que, dans les équations différentielles

$$\frac{dy_i}{dx}=f_i(x, y_1, \ldots, y_n) \quad (i=1, \ldots, n),$$

les fonctions f_i sont continues dans le domaine

$$G: \quad 0\leqq x\leqq a, \ |y_i|\leqq b \quad (i=1, \ldots, n),$$

et que l'on ait

$$f_i(x, 0, \ldots, 0)=0 \text{ pour } 0\leqq x\leqq a \quad (i=1, \ldots, n).$$

Alors pour que la solution des équations issue de l'origine $(0, 0, \ldots, 0)$, *O, soit unique, il faut et il suffit qu'il existe une fonction continue dans G, $\varphi(x, y_1, \ldots, y_n)$, telle que*

$$\varphi(x, 0, \ldots, 0)=0 \text{ pour } 0\leqq x\leqq a,$$

mais $\varphi(x, y_1, \ldots, y_n)>0$ *pour* $|y_1|+|y_2|+\ldots+|y_n|\neq 0$,

vérifiant dans G la condition de Lipschitz par rapport à $(y_1, \ldots, y_n)$, c'est-à-dire que, K étant une constante,

$$|\varphi(x, y_1, \ldots, y_n)-\varphi(x, \bar{y}_1, \ldots, \bar{y}_n)|\leqq K(|y_1-\bar{y}_1|+\ldots+|y_n-\bar{y}_n|),$$

pour deux points quelconques de G, $(x, y_1, \ldots, y_n)$ et $(x, \bar{y}_1, \ldots, \bar{y}_n)$, et que, pour tout point $(x, y_1, \ldots, y_n)$ intérieur de G, on ait

$$\overline{\lim_{t\to 0}}\frac{1}{t}\{\varphi[x+t, y_1+tf_1(x, y_1, \ldots, y_n), \ldots, y_n+tf_n(x, y_1, \ldots, y_n)]$$
$$-\varphi(x, y_1, \ldots, y_n)\}\leqq 0.$$

Si, en effet, la solution est unique, en posant

$$\varphi(x, y_1, \ldots, y_n)=D(O, P), \quad P: (x, y_1, \ldots, y_n),$$

la fonction φ satisfait à ces conditions ; car, au cas actuel, $D(O, P)$ est une fonction continue de P, non négative, s'annulant exclusivement sur la courbe solution issue de O, c'est-à-dire l'axe des x, et vérifiant l'inégalité, pour deux points P et Q sur un même hyperplan perpendiculaire à l'axe des x,

$$|D(O, P)-D(O, Q)|\leqq D(P, Q)=\overline{PQ},$$

et la fonction $D(O, P)$ ne peut pas croître avec x sur une courbe solution des équations différentielles parce que, si P_1 et P_2 sont deux points sur une même courbe solution, P_2 à droite de P_1, on a

$$D(O, P_2)\leqq D(O, P_1)+D(P_1, P_2) \text{ où } D(P_1, P_2)=0.$$

Inversement, s'il existe une telle fonction φ, on a, avec une solution issue de O, $y_i=y_i(x)$ $(i=1, \ldots, n)$, une fonction de x, $\varphi[x, y_1(x), \ldots, y_n(x)]$, qui s'annule pour $x=0$ et dont les nombres dérivés ne sont jamais positifs, et par suite cette fonction ne peut pas être positive ; cela entraîne que

$$y_i(x)=0 \text{ pour } 0\leqq x\leqq a \quad (i=1, \ldots, n). \quad C.\ q.\ f.\ d.$$

Si l'on a

$$|f_i(x, y_1, \ldots, y_n)|\leqq L\sqrt{y_1^2+\ldots+y_n^2} \quad (i=1, \ldots, n), \quad L: \textit{const}^{te},$$

on vérifiera l'énoncé du théorème précédent en posant

$$\varphi(x, y_1, \ldots, y_n)=(y_1^2+\ldots+y_n^2)e^{-2nLx}.$$

C'est le cas ordinaire de la condition de Lipschitz.

Remarquons enfin d'après le Mémoire antérieur de l'auteur (*loc. cit.*) que, pour le cas d'une seule équation, on peut disposer la fonction φ à posséder ses dérivées partielles du premier ordre continues ; on a, en effet, le théorème suivant :

Soit une équation différentielle

$$\frac{dy}{dx}=f(x, y),$$

où $f(x, y)$ est une fonction continue dans le domaine D $[0\leqq x\leqq a, 0\leqq y\leqq b]$, telle que $f(x, 0)=0$ pour $0\leqq x\leqq a$. Si la solution issue de l'origine $(0, 0)$ est unique, il existe une fonction $\varphi(x, y)$ continue dans D, ainsi que ses dérivées $\varphi'_x(x, y)$ et $\varphi'_y(x, y)$, telle que

$$\varphi(x, 0)=0, \quad \varphi'_y(x, 0)=0$$

mais $\varphi'_y(x, y) > 0$, $\varphi'_x(x, y) + \varphi'_y(x, y) f(x, y) < 0$ *pour* $y > 0$.[1]
La réciproque est déjà évidente.

L'existence d'une telle fonction $\varphi(x, y)$, dépourvue seulement des dérivées partielles aux points de l'axe des x, a été le résultat antérieur. Pour combler cette lacune, on n'a, si les conditions n'étaient pas réalisées, qu'à prendre une autre fonction $\Phi[\varphi(x, y)]$ au lieu de φ, $\Phi(z)$ étant une fonction continue, ainsi que sa dérivée, pour $z \geqq 0$, telle que

$$\Phi(0) = 0, \quad \Phi'(z) > 0 \text{ pour } z > 0$$

et

$$\Phi'(z) = o\left(\frac{1}{\max\limits_{\varphi = z}(|\varphi'_x|, \varphi'_y)}\right) \text{ pour } z \to +0.$$

Pour un système d'équations différentielles nous ne pouvons pas encore établir un résultat analogue.[2]

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1. On peut ajouter, de plus, la condition que la fonction

$$\frac{\varphi'_x(x, y)}{\varphi'_y(x, y)}$$

soit continue dans D et égale à 0 sur l'axe des x. C'est par le Théorème 1 et le Lemme du Mémoire cité de l'auteur.

2. *Note ajoutée à la correction des épreuves.*—Après la communication de ce Mémoire à la rédaction, je suis parvenu à compléter le théorème d'unicité au début du § III en démontrant l'existence de la fonction $\varphi(x, y_1, \ldots, y_n)$ ($\geqq 0$ selon que $|y_1| + \cdots\cdots + |y_n| \geqq 0$) *douée des dérivées premières continues* $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y_1}, \ldots, \frac{\partial \varphi}{\partial y_n}$ *dans* G (*la frontière incluse*) *satisfaisant à l'inégalité*

$$\frac{\partial \varphi(x, y_1, \ldots, y_n)}{\partial x} + \sum_{i=1}^{n} \frac{\partial \varphi(x, y_1, \ldots, y_n)}{\partial y_i} f_i(x, y_1, \ldots, y_n) \leqq 0.$$

Pour cela, il suffit de remplacer seulement la fonction φ obtenue dans le texte par sa moyenne intégrale convenablement prise. J'ai l'intention de pousser ces manières de raisonnements, dans un article prochain, sur un problème lié concernant l'unicité en tout point initial du domaine de définition d'un système différentiel.

On the Manner of Molecular Grouping in Glass

By Usaburo Yoshida

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Abstract

The idea of cybotaxis as affecting liquid and super-cooled liquid was extended to the case of glass, and various physical properties of glass such as viscosity, specific heat, thermal expansion, electrolytic conduction and mechanical properties were examined from the standpoint of this idea. For plastic deformation of metals and the sintering of powder crystals of metals etc., an explanation based on this idea has been given. The fracture of glass at its elastic limit is compared with the plastic deformation of metals.

Introduction

Recently the writer[1] proposed a hypothesis concerning the internal constitution of liquid, super-cooled liquid and amorphous solid, by assuming the presence of cybotactic groups or cybotactic crystals in these states of matter. A cybotactic group, the writer considered, is formed spontaneously in the liquid, by the aggregation of a few liquid molecules in a certain manner of configuration. The manner of molecular configuration will depend upon the number of constituent molecules ; and when this number is more than necessary to construct a unit cell of the crystal of the substance, the manner of molecular configuration will be the same as that of the crystal. This kind of cybotactic groups was called by the writer the " cybotactic crystal ". The presence of such cybotactic crystals and cybotactic groups in liquid, in super-cooled liquid and in amorphous solid was assumed by the writer, to conform with the results obtained with the diffraction experiments with x-rays and electron waves. The cybotactic groups in a liquid (including super-cooled state) at the temperatures higher than the solidifying temperature of the substance are of only momentary existence. They are formed spontaneously here and there in the liquid, and immediately decompose into free liquid molecules ; and statistical equilibrium between the cybotactic groups and the free liquid molecules is attained when the number of molecules forming

1. U. Yoshida : These Memoirs, **23**, 207 (1941).

the cybotactic groups in unit time becomes equal to that decomposing into the free liquid molecules in the same time. As the molecular aggregation into the cybotactic groups is caused by the attractive force between molecules, their decomposition into free liquid molecules is brought about by overcoming the potential energy of the intermolecular attraction with the aid of the kinetic energies of the constituent molecules due to thermal agitation. Thus the decrease of temperature increases the chance of formation and decreases that of decomposition; and consequently the proportion of the total amount of the cybotactic groups to that of the free molecules in equilibrium will increase gradually with decreasing temperature. When the temperature is continuously so much reduced that it is far below the melting point of the substance, the total amount of the cybotactic groups will be increased correspondingly, and the substance will at last come to the state in which the major part of its molecules are in the state of cybotactic combination. This extreme case of the super-cooling of a substance was considered by the writer to be its glassy state or its state of amorphous solid. An enormous increase in the viscosity of the substance in transforming from the liquid to the amorphous solid state was hence considered by the writer as mostly due to the enormous increase of the cybotactic groups with decreasing temperature. In the following various physical properties of glass were investigated from the viewpoint of cybotaxis as simply stated above.

Viscosity

A high degree of viscosity of a glass in its annealing temperature range requires considerable time to reach the constant value characteristic of a given temperature when it is maintained at this temperature; when it is higher initially it decreases gradually and when it is lower initially it increases gradually to the same constant value. This time required to reach the constant value decreases with increasing temperature. If the constituent molecules were entirely free from each other and had not the property to form molecular aggregation or chemical change, the equilibrium value of the viscosity characteristic of a given temperature would be reached immediately as soon as that temperature was attained by the whole mass of the substance. Consequently the presence of considerable time-lag in reaching the equilibrium value of the viscosity seems necessarily to imply the occurrence of a change in the state of molecular aggregation or in chemical

combination in the process of heat treatment. As the latter, among the effects of these two causes, is not probable to take place always for varions kinds of glasses at temperatures lower than the annealing temperature range, the presence of an immense number of cybotactic groups of considerable size seems to be necessary to explain the viscous property of glass at comparatively low temperature. Other than the viscosity, most of the physical properties of glass such as thermal expansion, specific heat, optical property and electrical conduction etc. are accompanied, in heat treatment, by the time-lag or the hysteresis of considerable amount at a comparatively low temperature, These facts seem to give powerful support to the cybotactic hypothesis. The process of annealing, when looked upon from the view-point of this hypothesis, is simply minimizing the nonuniformity in cybotactic states at different portions of the specimen by bringing these different states to the same one of equilibrium peculiar to the annealing temperature.

As was done in the former report, let x be the number of constituent molecules in a cybotactic crystal, n the number at a certain temperature of the cybotactic crystals in unit volume and N the total number of molecules in unit volume including both free and constituent ones of the cybotactic crystal; then the total number of free molecules in unit volume will be $N-xn$. Among these a certain fraction α per unit time will be aggregated to form the cybotactic crystals, and at the same time a certain fraction β of the cybotactic crystals per unit time will decompose into free molecules. Thus, in equilibrium, we have

$$\alpha(N-xn)=x\beta n, \quad \cdots\cdots(1)$$

$$\therefore \quad n=\frac{1}{1+\frac{\beta}{\alpha}}\frac{N}{x} \quad \cdots\cdots(2)$$

For a certain kind of cybotactic crystal in a certain liquid, both α and β will depend only upon the temperature. With increase of temperature, except when it is very low, α will decrease and β will increase continuously in consequence of the increase of the kinetic energy of molecules; and thus n will decrease continuously with increasing temperature. However, with the glassy state of a substance, n should be considered to be very large, and the temperature dependence of α and β will be different in this case as is schematically represented by the curves drawn in full lines in Fig. 1. At the temperatute T_m, which may be looked for as the melting point of the cybotactic crystals, the

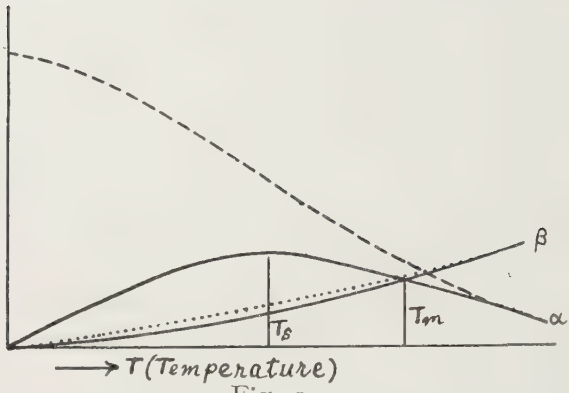

Fig. 1

values of α and β are equal; and half the total number of the molecules comes to be aggregated as cybotactic crystals. Below this temperature the number of molecules aggregated as cybotactic crystals increases with decreasing temperature, and the general aspect of the distribution of free molecules in narrow interstices between neighbouring cybotactic crystals will be such as shown by Fig. 2. In this state the stability or the average duration of persistence of the cybotactic crystals will be considerably increased due to the want of kinetic energies of molecules which are necessary to decompose the crystals. Moreover this stability will be still more increased by the difficulty of expansion accompanying the decomposition of the cybotactic crystals against an enormous pressure which is supposed to be acting in the glassy state. Consequently the value of β will become smaller than in the ideal case where no remarkable pressure is acting against the expansion by decomposition of the cybotactic crystals. This ideal case is shown by the curve drawn in dotted line in Fig. 1. Next the rate of aggregation of free molecules into cybotactic crystals will be very much reduced, as compared with the case when the majority of molecules are free, by the lack of neighbouring and adjoining free molecules and also by the absence of free spaces to be occupied by the formation of individual cybotactic crystals. Because of these circumstances, α will not increase continuously with decreasing temperature as is shown by the broken line in Fig. 1, which corresponds to the ideal case containing no cybotactic groups in the substance; but after passing through a maximum, it will decrease again with decrease of the temperature. This tendency of lessening the value of α at low temperature will be much more remarkable in multi-component glass. In this case we must expect the presence of the cybotactic crystals of different composition, and every individual kind of cybotactic crystal is formed by the aggregation of a certain number of

Fig. 2

molecules belonging to certain kinds of components. The chance for this to occur will be much smaller than in the case of one-component glass. The low values of both α and β at low temperatures mean that the stability or the duration of persistence of the molecules both in the free state and in the aggregated state as cybotactic crystals is great: i. e. that it requires a long time for the change to take place from the free state to the aggregated one or vice versa. Thus when temperature is changed, new equilibrium is only attained at low temperatures after a time-lag of considerable duration as is actually observed with viscosity and the other physical properties of glass.

In the case of annealing glass, the lower limit of the annealing temperature range was called "the strain point" by Littleton,[1] and was defined as that temperature from which a piece of glass can be quickly cooled without introducing permanent strain. If our consideration is correct that α has a maximum at a certain temperature as is shown in Fig. 1, then the temperature dependence of β/α will be such as shown in Fig. 3. The value of β/α will increase only slightly up to the temperature T_s, which is not much different from that corresponding to the maximum value of α, and then rapidly with increasing temperature. Corresponding to such slight change of β/α below T_s, the equilibrium number of the cybotactic crystals per unit volume at the temperature T_s will be almost the same as that at room temperature. Thus when the glass, after being kept for a long time at the temperature T_s, is cooled quickly to the room temperature, no remarkable change in its internal structure will be called forth throughout the whole portion of the specimen by this sudden change of temperature.

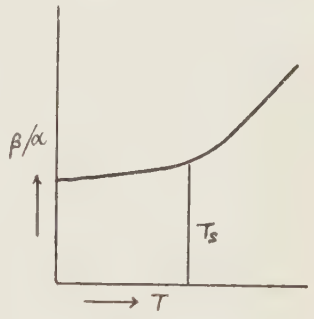

Fig. 3

Though the change in the value of β/α is very slight below the temperature T_s, it will not be entirely free from temperature change; and a slight decrease with decreasing temperature is expected even below T_s. Corresponding to this slight decrease in β/α a slight increase in the equilibrium number n of the cybotactic crystals accompanies decreasing temperature, indicating that the change in the equi-

1. Morey: Properties of Glass p. 174 (1938).

librium state in the internal structure is contraction with decreasing temperature. When glass in structural equilibrium at high temperature is cooled to room temperature, it requires a considerable time to attain the contracted state which is in structural equilibrium at room temperature, as is revealed by the "secular rise of the zero point" of a newly made thermometer. This secular rise of the ice point can be temporarily depressed by giving the thermometer glass a slight structural expansion by heating the thermometer to 100°C and cooling it quickly.

Specific Heat

Thomas and Parks[1] measured the specific heat of boric oxide glass with continuously changing temperature by means of a radiation calorimeter. The sample was prepared by heating it in a platinum crucible for 8 hours at 900°C., and pouring it into a silver calorimeter, kept at 500°C, then cooled quickly to 50°C in about 30 minutes. The specific heat of this quickly cooled specimen was then measured in a continuous series of observations, from 33°C to 345°C, in about 14 hours. The glass was then cooled slowly in 18 hours, and a second series of specific heat measurements was made during the cooling from 332°C to 112°C. The specific heat of this slowly cooled glass was again measured by heating it gradually from 35°C to 325°C. The results of these measurements are assembled in Fig. 4. There was a rapid increase in specific heat in the softening region of the sample.

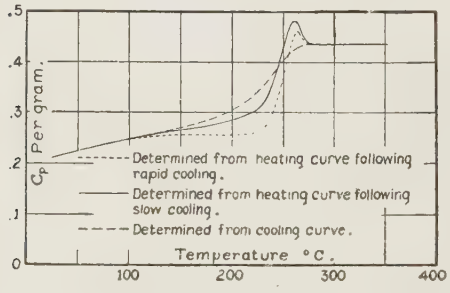

Fig. 4

From the view-point of the writer's hypothesis the explanation for this rapid increase is simple: it is due to the latent heat of fusion of the cybotactic crystals whose number decreases remarkably in the softening region by raising the temprature. The differences between the three curves in Fig. 4 arise from the lag or the hysteresis in the cybotactic change of the sample, just as in the case of viscosity. The specific heat determined from the heating curve following rapid cooling is smaller than that obtained from

1. S. B. Thomas and G. S. Parks: J. Phys. Chem., **35**, 2091 (1931).

the heating curve following slow cooling. This is caused by a smaller content of the cybotactic crystals in the former case by virtue of the fact that the time elapsed during the cooling was very much shorter than in the latter, so that the departure from the cybotactic equilibrium at the cooled temperature was much more remarkable in the former case. The two curves obtained on heating show a hump just before flattening out. This is also due to the hysteresis in the cybotactic change and is caused by the release of the stored-up lag. The aspect of the hysteresis in the cybotactic change will be clearly understood if we compare the curve obtained from the cooling curve with that obtained from the heating curve following the preceding slow cooling. On heating curve the specific heat is smaller at lower temperatures by virtue of the lag in the cybotactic change; and then the stored-up lag is restored at a narrow range of higher temperatures by forming a hump in the specific heat curves. All the specific heat curves coincide entirely at temperatures higher than about 275°C. From the writer's view-point this is due to the fact that the lag in the cybotactic change is unnoticeable at these temperatures. Three specific heat curves coincide also at temperatures lower than about 100°C. At these temperatures, the content of the cybotactic crystals is supposed not to be the same. But the lag in the cybotactic change will be enormous in this case, and no noticeable change in the amount of the cybotactic crystals on heating or on cooling will take place at these low temperatures in any case.

Thermal Expansion

It is known that the glass expands almost linearly up to the lower critical point or the transition point, and then more rapidly with increasing temperature. It seems to the writer that the amount of the free molecules formed by decomposition of the cybotactic crystals becomes especially remarkable above this point; and that the more remarkable expansion above this point is due to the relaxation of the tight internal structure, caused by the increase in number of free molecules and by sufficient increase in kinetic energies of the molecules at this temperature. According to the experiment made by Turner and Winks,[1] glass chilled quickly from high temperature behaves abnormally about this point as is shown by the lowest curve in Fig. 5. At a temperature somewhat below this point its expansion lessens

1. Morey: The Properties of Glass, p. 266 (1938).

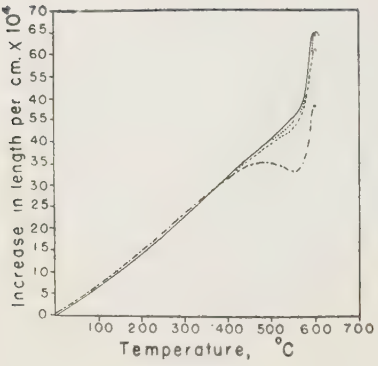

Fig. 5

remarkably when compared with that observed with annealed specimens, and again increases from the critical point. With this glass, the initial cybotactic state will be much nearer than the annealed one to that at high temperature from which it is chilled quickly, and the content of the cybotactic crystals will be much less at lower temperatures than that corresponding to the equilibrium value. When the temperature is raised to the value somewhat below the critical point, the equilibrium state will be approached by an increase in number of the cybotactic crystals rather than the decrease, by bringing forth the abnormal decrease in its thermal expansion. Such abnormal decrease in expansion coefficient can be made to occur at a certain temperature more abruptly by choosing the appropriate rate of temperature increase and the thermal history of the specimen, and this temperature is sometimes called the "transformation point". Littleton[1] and Morey[2] consider that the effect observed is the result of failure of the glass to reach internal equilibrium at lower temperature with the result that, when the viscosity has been sufficiently diminished by heating, the frozen-in lag is suddenly released. Their view agrees entirely with the writer's in the sense that by the internal equilibrium the equilibrium in the cybotactic state is meant.

Electrolytic Conduction

It is well known that at higher temperatures ordinary glasses conduct electricity electrolytically. With proper experimental procedure we can even convey the Na ions from one side of soda glass to the other. The temperature dependence of the specific resistance of glasses has been measured by many authors; and it is known that the specific resistance runs, with temperature change, almost parallel, though not proportionally, with the coefficient of viscosity: i. e. that the specific conductivity of a glass changes, almost parallel, with its fluidity with temperature change. According to the cybotactic theory,

1. Morey: The Properties of Glass, p. 267 (1938).
2. Morey: The Properties of Glass, p. 267 (1938).

the increase in fluidity of glass is due mostly to the increase in number of the free molecules in proportion to that of the cybotactic groups. Thus if we follow the same reasoning, the electrolytic conductivity of glass must be ascribed mostly to the transport of the ions along the interstices of cybotactic crystals, where the molecules and the ions are free to move at higher temperatures. As is evident from the cybotactic idea an individual cybotactic crystal has no permanent stability. It is formed spontaneously somewhere in the substance, and is of only a momentary existence at higher temperatures. Consequently the paths of the ions are not fixed, and they will change from moment to moment by chance. The transportation of the ions of considerable amount at temperatures not very high, where the glass is still in a solid state, seems to be explained by the presence of this circumstance; without which the transport of the ions would be very difficult if they were confined permanently along very narrow and irregular interstices between neighbouring cybotactic crystals.

Littleton and Wetmore[1] measured at various temperatures the change of resistivity with time of annealed and chilled glasses having the same composition, and found that the specific resistivity of the chilled glass increased with heating time; and that the specific resistivity of the annealed one, which was greater initially than that of the chilled glass, approaches the value of the chilled one as the time of heating is increased. This is a hysteresis effect in the cybotactic state, and is due to the fact that the initial amount of the cybotactic crystals in the chilled glass corresponding to the condition of the cybotactic equilibrium at a temperature higher than that of the experiment was smaller than that of the equilibrium state at the temperature of the experiment; whereas that of the annealed glass was initially greater than that of the chilled one corresponding to the equilibrium value at lower temperature.

Elastic Property and Fracture at Low Temperature, and Fluidity at High Temperature

The elasticity of glass at room temperature is almost perfect below its limit. Plastic bending or plastic flow at room temperature can be detected only when the glass is subjected continuously for a sufficiently long time to the deforming force. Moreover it is known that

1. Morey, p. 462.

the internal stress existing in a glass can hardly be removed at room temperature however long it may be kept at that temperature. From the point of view of cybotaxis as was stated before, these facts indicate that the cybotactic exchange between the cybotactic crystals and the free molecules is almost, though not entirely, stopped at this temperature. In elongation testing, when the load is increased continuously the glass breaks just at its elastic limit, without showing the presence of the plastic deformation as is commonly observed with metals and other substances. Glass may be broken in many different ways, by tension, compression, twisting, or impact etc. It is probable, however, that in all the various methods of applying stress to glass, fracture always takes place in tension. The works of Griffith[1] and of Joffé[2] make probable the hypothesis that since the true ultimate strength in tension is that determined by the molecular cohesion the far lower values found in practice are caused by surface scratches or flaws. Griffith assumed that the molecules in the glass orient themselves in groups such that in certain directions the attractive forces are minimum, and that such groupings are incipient flaws. A similar supposition to Griffith may also be given from the stand point of the cybotactic hypothesis. As the distribution of the cybotactic crystals is entirely at random, the interstices between neighbouring cybotactic crystals will be of various widths and of various densities of free molecules, and consequently will be the fields of cohesive forces of various strengths. If so, the incipient flaws or fissures will be formed along the surfaces obtained by connecting the interstices of weakest cohesive force in the manner shown by the line A B in Fig. 6.

Fig. 6

In the case of a metal, which is composed of micro-crystals, plastic deformation occurs beyond its elastic limit. This plastic deformation is considered to be due to the slip along certain atomic planes of the crystal. In doing this the portion of the micro-crystals where the slip has occurred will be broken down into very fine crystals even of the cybotactic size and will be distorted in their lattices by the expenditure of considerable amount of the applied work ; and the local

1. Griffith: Trans. Roy. Soc., London, A, **221**, 162 (1920).
2. Joffé: The Physics of Crystals (1928).

heating along the slipped plane will be sufficient to melt the smashed crystals, owing to the fact that their melting point is much lower than that of the adult crystals. By subsequent rapid cooling, which takes place almost at the same time as the local heating, the melt will be brought to the super-cooled state immediately. Thus after the deformation, the defects which have been caused by deformation will be almost remedied, though not perfectly, by being cemented by the solidified super-cooled amorphous layer formed there, and the specimen becomes able to sustain again the applied stress in an elastic manner.

In the case of glass, since it is not, like metals, composed of microcrystals excepting those of cybotactic size, it cannot be deformed by means of slipping along some atomic plane. Thus when tension of sufficient strength is applied to a part of the glass in a certain direction, most of the deformation will take place by the separation of the two adjacent portions which are in contact at the surface of the incipient or embryonic flaws extending nearly perpendicularly to the direction of the applied tension; and no remarkable local heating, subsequent melting and solidification as in the case of metals, which is necessary to remedy the widely opened flaws, can be expected in the case of the glass. This seems to be the reason that the glass can not be deformed plastically beyond its elastic limit at room temperature and that it breaks abruptly at this limit. If the cybotactic exchange between the cybotactic crystals and the free molecules previously noted should be of remarkable degree, this would also take a part in remedying the separated flaws; but this seems to be of such minor degree at room temperature as to have escaped our detection as is stated before. With respect to this point Spencer's experiment[1] is worthy of mention. "He wrapped fibres of glass around a cylinder two centimeters in diameter, and held them for months. On cutting the fibres loose they immediately opend out to coils of about 60 cms in diameter, and seemed to have acquired a permanent set. They were left floating on mercury and they gradually diminished in curvature, but after five years they were still slightly curved". Spencer's interesting experiment shows that the amount of cybotactic exchange was still very slight, if any, after storage for months at room temperature, and that the deformation impelled by coiling is almost recovered by the shrinkage of the flaws opened widely by coiling.

1. Moyey: The Properties of Glass, p. 318 (1938).

In the case of abrasion and polishing of the surface of a glass the circumstances are different. In this cese local heating of such a degree as to cause a surface flow by softening its thin layer can be expected just as in the case of metals.

When the temperature of a glass is raised considerably from room temperature the values of both α and β will increase correspondingly, and consequently the cybotactic exchange becomes very active, so that the cybotactic equilibrium can be attained much more quickly than before; and at the same time the number of the cybotactic crystals will diminish remarkably. In this state the glass is endowed with fluidity. Different parts of it can be displaced easily from each other, and separate pieces of it can coalesce by being pressed or merely brought in contact with each other. In relation to the fluidity of glass, the sintering of crystal powders at high temperatures is worth consideration. Jeffries and Archer[1] state: "It has been erroneously stated that metal powders will not sinter to form a coherent mass when heated unless the pressure is sufficient to cause fusion at the temperature used. As a matter of fact, tungsten powder can be sintered below the melting point with no pressure except that due to its own weight. In fact, it is very difficult to see how pressing a metal powder briquet could have any effect on the melting point except to raise it". In ordinary sense the melting point of a crystalline substance signifies the equilibrium temperature at which the adult crystal coexists with its melt as is stated in a previous paper.[2] In this sense the melting point of a metal will not be much affected by pressure, and it will be rather raised by pressure with a metal such as tungsten as is stated by Jeffries and Archer. However if we accept the view that, when the crystals are smashed to extreme fineness comparable with the cybotactic crystals as is considered by the writer, their melting point or their stabibity is very much reduced, coalescense of crystal powders under pressure at high temperature will be easily understood just as in the case of the plastic deformation previously explained. The circumstance that the sintering of powder crystals is done usually at very high temperature greatly facilitates the melting of the finely divided crystals, and consequently in giving fluidity to the briquet of the powder crystals to form a coherent mass.

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1. Jeffries and Archer: The Science of Metals, p. 132 (1924).
2. Yoshida; These Memoirs, A, **23**, 207 (1941).

Ueber die Einteilung des Simplexes

Von **Toshizô Matsumoto**

(Eingegangen am 3ten, September, 1940)

Es ist gut gesagt dass ein Simplex in kleineren Simplexen von Durchmesser minder als eine gegebene positive Zahl deren zwei nur einer Seite (Fläche, Kante oder Ecke) benachbart werden können zerlegen lässt.[1] Da wir mit höheren Räumen handeln, besser ist es analytisch angeben. In dieser Note wollen wir ein Art simplizialer Zerlegung analytisch angeben, wobei die kleinere Simplexen höchstens halbierten Durchmessern haben. Widerholt man diese Verfahren, so gelangt er an kleineren Simplexen von Durchmessern kleiner als eine gegebene positive Zahl.

Im folgenden schreiben wir nur einen Buchstaben (x) an Stelle der rechtwinkligen Koordinaten $(x_1, x_2, \ldots\ldots, x_n)$ des n-dimensionalen Euklidischen Raumes. Ein Simplex T_n hat $n+1$ Ecken. Seien sie $P_1(x^1), P_2(x^2), \ldots\ldots, P_{n+1}(x^{n+1})$. Da diese Koordinaten linear unabhängig (mit positiven Koefficienten) sind, lässt sich ein beliebiger Punkt (ξ) des T_n folgendermassen darstellen:

$$\begin{aligned} &\xi \equiv \lambda_1 x^1 + \lambda_2 x^2 + \ldots\ldots + \lambda_{n+1} x^{n+1}, \\ &\lambda_1 + \lambda_2 + \ldots\ldots\ldots + \lambda_{n+1} = 1, \\ &\lambda_1, \lambda_2, \ldots\ldots, \lambda_n \geqslant 0. \end{aligned} \qquad (1)$$

Verschwinden nicht alle λ, so bedeutet (1) den inneren Punkt des Simplexes. Wenn aber mindestens ein λ verschwindet, so ist (1) ein Punkt der Seite (die Begrenzung der Punktmenge T_n) welche selbst auch ein Simplex von höchstens $n-1$ Dimensionen ist.

Sei z. B, $\lambda_{n+1}=0$, dann ist die Menge der Punkte:

$$\begin{aligned} &\sigma \equiv l_1 x^1 + l_2 x^2 + \ldots\ldots + l_n x^n, \\ &l_1 + l_2 + \ldots\ldots + l_n = 1, \\ &l_1, l_2, \ldots\ldots, l_n \geqslant 0 \end{aligned} \qquad (2)$$

ist ein $(n-1)$-dimensionales Simplex S_{n-1} mit $P_1, P_2, \ldots\ldots, P_n$ als Eckpunkten. Es ist leicht beweisbar, dass die Menge der gesammten Punkte auf den Strecken, welche einen beliebigen Punkt von (2) mit dem Punkt $P_{n+1}(x^{n+1})$ verbindet, mit dem gegebenen Simplex T_n identisch ist. So nennen wir P_{n+1} den *Scheitelpunkt* und S_{n-1} das *Grundsimplex*, wobei T_n eine *Pyramide* darstellt.

1. Z. B. S. Stoïlow, Leçons sur les principe topologique (1938) p. 19.

Bezeichnet man den Mittelpunkt der Kante P_iP_j mit $P_{ij}(x^{ij})$ $i,j=1,2,\ldots\ldots,n+1$. Dann ergibt sich:

$$P_{ij}\equiv P_{ji},\qquad x^{ij}=x^{ji}=\frac{x^i+x^j}{2},\tag{3}$$

Nun aus T_n schneiden wir die Simplexe

$$t_n^i:\ P_iP_{i1}P_{i2}\ldots\ldots P_{i,i-1}P_{i,i+1}\ldots\ldots P_{i,n+1},\tag{4}$$
$$i=1,2,\ldots\ldots,n+1.$$

Den restbleibenden Körper nennen wir K_n. der ein konvexer Körper unseres Raumes ist. So haben wir

$$T_n=K_n+t_n^1+t_n^2+\ldots\ldots+t_n^{n+1}.\tag{5}$$

Nun wollen wir beweisen, dass die Koordinaten der Punkte von K_n sind

$$\begin{aligned}x&\equiv l_{12}x^{12}+l_{13}x^{13}+\ldots\ldots+l_{n,n+1}x^{n,n+1}\\&=\sum_{i<j}^{n+1}l_{i,j}x^{i,j},\\&l_{12}+l_{13}+\ldots\ldots+l_{n,n+1}=1,\\&l_{12},l_{13},\ldots\ldots,l_{n,n+1}\geqq 0;\end{aligned}\tag{6}$$

zunächst aber bedeutet K_n die Menge der durch (6) gegebene Punkte x. Seien z. B. η die Koordinaten der Punkten von t_n^1, so hat man

$$\begin{aligned}\eta&=\mu_1x^1+\mu_2x^{12}+\ldots\ldots+\mu_{n+1}x^{1,n+1},\\&\mu_1+\mu_2+\ldots\ldots+\mu_{n+1}=1,\\&\mu_1,\mu_2,\ldots\ldots,\mu_{n+1}\geqq 0.\end{aligned}\tag{7}$$

Die Werte (3) eingesetzt, haben wir

$$\eta=\frac{x^1}{2}+\frac{1}{2}(\mu_1x^1+\mu_2x^2+\ldots\ldots\mu_{n+1}x^{n+1}),$$

auch
$$\begin{aligned}x=&\frac{1}{2}(l_{12}+l_{13}+\ldots\ldots+l_{1,n+1})x^1\\&+\frac{1}{2}(l_{21}+l_{23}+\ldots\ldots+l_{2,n+1})x^2\\&+\ldots\ldots\ldots\ldots\ldots\ldots\\&+\frac{1}{2}(l_{n+1,1}+l_{n+1,2}+\ldots\ldots+l_{n+1,n})x^{n+1},\end{aligned}$$

$$l_{i,j}=l_{j,i}.$$

Man denke den Punkt (1):

$$\xi=\lambda_1x^1+\lambda_2x^2+\ldots\ldots+\lambda_{n+1}x^{n+1}.$$

I. Wenn $\lambda_1>\frac{1}{2}$, so gehört ξ zum t_n^1. So können wir annehmen:

$$\lambda_1=\frac{1+\mu_1}{2},\quad \lambda_2=\frac{\mu_2}{2},\ \ldots\ldots,\ \lambda_{n+1}=\frac{\mu_{n+1}}{2}.$$

Da $\mu_1 > 0,\ \lambda_2, \ldots\ldots, \lambda_{n+1} < \frac{1}{2}$,

hat man $0 \leqslant \mu_2, \ldots\ldots, \mu_{n+1} < 1$,

und $$\frac{1+\mu_1+\ldots\ldots+\mu_{n+1}}{2} = 1\ ;$$

folglich $\mu_1+\mu_2+\ldots\ldots+\mu_{n+1}=1$.

Noch der Punkt kann nicht zu K_n gehören, denn wenn anders

$$\frac{1}{2}(l_{12}+l_{13}+\ldots\ldots+l_{1,n+1}) > \frac{1}{2}.$$

sein müsste, was unmöglich ist.

II. Wenn $\lambda_1 = \frac{1}{2}$, so gehört ξ sowie dem t_n^1 auch dem K_n an. In diesem Falle können wir setzen:

$$\mu_1 = 0,\ \lambda_2 = \frac{\mu_2}{2},\ \ldots\ldots,\ \lambda_{n+1} = \frac{\mu_{n+1}}{2}\ ;$$

folglich $\mu_1+\mu_2+\ldots\ldots+\mu_{n+1}=1$.

So haben wir

$$\begin{aligned}\eta &= \frac{x^1}{2} + \frac{1}{2}(\mu_2 x^2 + \ldots\ldots + \mu_{n+1}x^{n+1}) \\ &= \mu_2 \frac{x^1+x^2}{2} + \ldots\ldots + \mu_{n+1}\frac{x^1+x^{n+1}}{2} \\ &= \mu_2 x^{12} + \mu_3 x^{13} + \ldots\ldots + \mu_{n+1}x^{1,n+1},\end{aligned}$$

so bilden diese Punkte das $(n-1)$-dimesionale Simplex U_{n-1}^1 mit den Eckpunkten $P_{12}, P_{13}, \ldots\ldots, P_{1,n+1}$. Das ist das Grundsimplex der Pyramiden t_n^1 mit P_1 als seinen Scheitelpunkt.

Nunmehr zeigen wir, dass diese Punkte ξ auch zu K_n gehören. Für das genügt es nur so nehmen dass

$$\lambda_2 = l_{12},\ \lambda_3 = l_{13},\ \ldots\ldots,\ \lambda_{n+1} = l_{1,n+1}$$

und $$l_{i,j} = 0,\ \begin{matrix} i<j \\ i,j=2,\ldots\ldots,n+1 \end{matrix}.$$

Dann $$\begin{aligned}\xi &= \frac{1}{2}x_1 + \lambda_2 x^2 + \ldots\ldots + \lambda_{n+1}x^{n+1} \\ &= \frac{1}{2}(l_{12}+l_{13}+\ldots\ldots+l_{1,n+1})x^1 \\ &\quad + \frac{1}{2}l_{21}x^2 + \ldots\ldots + \frac{1}{2}l_{n+1,1}x^{n+1} \\ &= l_{12}x^{12} + l_{13}x^{13} + \ldots\ldots + l_{1,n+1}x^{1,n+1} \\ &= x.\end{aligned}$$

Daher haben das Simplex t_n^1 und der konvexe Körper K_n nur den Grundsmplex U_{n-1}^1 gemein.

III. Wenn $\lambda_i < \frac{1}{2} (i = 1, 2, \ldots, n)$, so gehört ξ zum Körper K_n allein.

Für $\xi = \varkappa$, haben wir die folgenden simultanen Gleichungen zu lösen:

$$\begin{cases} l_{12} + l_{13} + \ldots\ldots + l_{1,n+1} = 2\lambda_1, \\ l_{21} + l_{23} + \ldots\ldots + l_{2,n+1} = 2\lambda_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_{n+1,1} + l_{n+1,2} + \ldots\ldots + l_{n+1,n} = 2\lambda_{n+1} \end{cases} \tag{8}$$

wo
$$\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n = 1,$$
$$\lambda_1, \lambda_2, \ldots\ldots, \lambda_n \geqslant 0,$$
$$l_{12} + l_{13} + \ldots\ldots + l_{n,n+1} = 1,$$
$$0 \leqslant l_{i,j} = l_{j,i} \leqslant 1.$$

Zunächst sei $n = 2$; so haben wir die folgenden Gleichungen zu lösen:

$$\begin{cases} l_{12} + l_{13} = 2\lambda_1, \\ l_{21} + l_{23} = 2\lambda_2, \\ l_{31} + l_{32} = 2\lambda_3 \end{cases}$$

wo $l_{ij} = l_{ji}$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$,

$$\lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}.$$

Wir können annehmen, dass

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3.$$

So hat man

$$l_{13} - l_{23} = 2(\lambda_1 - \lambda_2),$$

Daher
$$l_{21} - l_{31} = 2(\lambda_2 - \lambda_3).$$
$$l_{13} = l_{23} + 2(\lambda_1 - \lambda_2).$$

Andererseits

$$l_{12} = -l_{23} + 2\lambda_2.$$

Da
$$l_{12} + l_{13} + l_{23} = 1,$$

hat man
$$l_{23} + 2\lambda_1 = 1.$$

Daher
$$l_{23} = 1 - 2\lambda_1.$$

Folglich
$$l_{12} = 1 - 2\lambda_3,$$
$$l_{13} = 1 - 2\lambda_2.$$

Man bemerke, dass

$$0 < l_{12}, \; l_{13}, \; l_{23} < 1,$$

und deren Summe gleich 1 ist.

Nun betrachten wir die Gleichungen (8) in Hinsicht auf die mathematische Induktion.

Wir können so annehmen, dass

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots\ldots \geqslant \lambda_{n+1}.$$

Jetzt denken wir das System der Gleichungen

$$\begin{cases} l_{12}+l_{13}+\ldots\ldots+l_{1n}=2\lambda_1-l_{1,n+1}, \\ l_{21}+\quad l_{23}+\ldots\ldots+l_{2n}=2\lambda_2-l_{2,n+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_{n1}+l_{n2}+\ldots\ldots+l_{n,n-1}=2\lambda_n-l_{n,n+1}. \end{cases} \tag{9}$$

wo $l_{n+1,1}+l_{n+1,2}+\ldots\ldots+l_{n+1,n}=2\lambda_{n+1}$;

aber die l noch nicht bestimmt sind. Da

$$\frac{2\lambda_1-l_{1,n+1}}{2}+\frac{2\lambda_2-l_{2,n+1}}{2}+\ldots\ldots+\frac{2\lambda_n-l_{n,n+1}}{2}$$

$$=1-\lambda_{n+1}-\lambda_{n+1}=1-2\lambda_{n+1}>0.$$

Schreiben wir

$$\lambda'_i\equiv\frac{2\lambda_i-l_{i,n+1}}{2(1-2\lambda_{n+1})}, \qquad i,j=1,2,\ldots,n.$$

$$l_{ij}'\equiv\frac{l_{ij}}{1-2\lambda_{n+1}},$$

So haben wir das folgende System zu lösen:

$$\begin{cases} l_{12}'+l_{13}'+\ldots\ldots+l_{1n}'=2\lambda_1', \\ l_{21}'+\quad l_{23}'+\ldots\ldots+l_{2n}'=2\lambda_2', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ l_{n1}'+l_{n2}'+\ldots\ldots+l_{n,n-1}'=2\lambda_n'. \end{cases} \tag{10}$$

Um die mathematische Induktion zu anwenden, sollen $\lambda_1', \lambda_2', \ldots\ldots, \lambda_n'$ unsere Bedingung erfüllen, nämlich:

$$0\leqslant\lambda_i'<\frac{1}{2}, \quad i=1,2,\ldots,n$$

wo $\lambda_1'+\lambda_2'+\ldots\ldots+\lambda_n'=1.$

Für dass $\lambda_1'=\frac{2\lambda_1-l_{1,n+1}}{2(1-2\lambda_{n+1})}<\frac{1}{2},$

hat man $2\lambda_1-l_{1,n+1}<1-2\lambda_{n+1},$

oder

Gleicherweise

$$\begin{cases} \lambda_1+\lambda_{n+1}-\frac{1}{2}l_{1,n+1}<\frac{1}{2}. \\ \lambda_2+\lambda_{n+1}-\frac{1}{2}l_{2,n+1}<\frac{1}{2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_n+\lambda_{n+1}-\frac{1}{2}\lambda_{n,n+1}<\frac{1}{2}. \end{cases} \tag{11}$$

Es ist leicht sichtbar, dass für $n>3$, die folgende Ungleichung richtig ist:

$$\lambda_2+\lambda_{n+1}<\frac{1}{2}.$$

Wäre $\lambda_2+\lambda_{n+1}\geqslant\frac{1}{2}$,

dann ist $\lambda_1+\lambda_{n+1}\geqslant\frac{1}{2}$.

Diese Ungleichungen addiert erhalten wir

$$\lambda_1+\lambda_2+2\lambda_{n+1}\geqslant 1.$$

Oder $1-(\lambda_3+\ldots+\lambda_{n+1})+2\lambda_{n+1}\geqslant 1$.

A fortiori $1-(\overline{n+1}-2)\lambda_{n+1}+2\lambda_{n+1}\geqslant 1$.

Oder $4\lambda_{n+1}\geqslant(n+1)\lambda_{n+1}$.

Die Behauptung ist daher richtig wenn $n>3$, wenn nur $\lambda_{n+1}>0$. Wäre aber $\lambda_{n+1}=0$, dann ist natürlich die Behauptung richtig.

Nun für $n>3$, wäre $\lambda_1+\lambda_{n+1}>\frac{1}{2}$, dann sind die Ungleichungen (11) befriedigt, wenn man folgendermassen annimmt:

$$\begin{cases} l_{1,n+1}=2\lambda_{n+1}, \\ l_{2,n+1}=\ldots\ldots=l_{n,n+1}=0. \end{cases} \tag{12}$$

Untersuchen wir den Fall wo $n=3$. Wäre z. B.

$$\begin{cases} \lambda_1+\lambda_4>\frac{1}{2}, \\ \lambda_2+\lambda_4\geqslant\frac{1}{2}. \end{cases}$$

Dann ist $\lambda_1+\lambda_2+2\lambda_4>1$.

Folglich $\lambda_1+\lambda_2+\lambda_3+\lambda_4>1$,

welche unmöglich ist. Daher höchstens haben wir

$$\begin{cases} \lambda_1+\lambda_4=\frac{1}{2}, \\ \lambda_2+\lambda_4=\frac{1}{2}, \\ \lambda_3+\lambda_4<\frac{1}{2}; \end{cases}$$

oder

$$\begin{cases} \lambda_1+\lambda_4\geqslant\frac{1}{2}, \\ \lambda_2+\lambda_4<\frac{1}{2}, \\ \lambda_3+\lambda_4<\frac{1}{2}. \end{cases}$$

Im ersten Falle, ist die Ungleichung (11) befriedigt. Im zweiten Fall, dem wir schon begegnet sind, haben wir zu setzen

$$\begin{cases} l_{11} = 2\lambda_1, \\ l_{21} = l_{31} = 0 ; \end{cases} \tag{13}$$

und auch (11) ist befriedigt.

Wenn aber im Allgemeinen, $\lambda_1 + \lambda_{n+1} \leqslant \frac{1}{2}$, so können wir beliebige positive Zahlen für $l_{1,n+1}, \ldots\ldots, l_{n,n+1}$ angeben, nur mit der Einschränkung, dass

$$l_{1,n+1} + \ldots\ldots + l_{n,n+1} = 2\lambda_{n+1}.$$

In allen Fällen können wir (11) befriedigen. Mittelst der mathematischen Induktion, das System (10) aufgelöst erhalten wir

$$l_{ij} = l_{ij}'(1 - 2\lambda_{n+1}). \quad i, j = 1, 2, \ldots, n. \tag{14}$$

Überdies da

$$l_{12}' + l_{13}' + \ldots\ldots + l_{n-1,n}' = 1.$$

so haben wir

$$\begin{aligned} l_{12} + l_{13} + \ldots\ldots + l_{n,n+1} &= (l_{12}' + \ldots\ldots + l_{n-1,n}')(1 - 2\lambda_{n+1}) \\ &\quad + l_{1,n+1} + \ldots\ldots + l_{n,n+1} \\ &= (1 - 2\lambda_{n+1}) + 2\lambda_{n+1} = 1. \end{aligned}$$

Daher das System (14) und beguem genommene Lösungszahlen $l_{1,n+1}$, $\ldots\ldots$, $l_{n,n+1}$ geben das gesuchte Lösungssystem von (8).

Kehren wir zum Beweis des Fundamentalsatzes zurück. Der konvexe Körper ist begrenzt durch die folgenden $n+1$ Simplexe von $n-1$ Dimensionen:

$$\begin{aligned} U_{n-1}^i : \quad & P_{i1} \ldots P_{i,i-1} P_{i,i+1} \ldots\ldots P_{i,n+1}, \\ & i = 1, 2, \ldots\ldots, n+1 ; \end{aligned}$$

und auf jeder Seite von T_n liegenden konvexen Körpern von $n-1$ Dimensionen K_{n-1}^j, deren Anzahl gleich der Seite ist; d. h. $j = 1, 2, \ldots$, $(n+1)$. Der konvexe Körper K_{n-1}^j ist construiert aus dem Simplex mit Eckpunkten $P_1 \ldots P_{j-1} P_{j+1} \ldots\ldots P_{n+1}$, ganz wie K_n aus T_n.

Nun nehmen wir an dass diese konvexen Körper alle simpliziar eingeteilt sind. So bestehen alle Seiten von K_n aus $(n-1)$-dimensionalen Simplexen V_{n-1}^j. Es ist leicht zu sehen, dass je zwei dieser Simplexen nur eine Seitensimplex von niederer Dimension gemein haben können. Projizieren wir diese Simplexe U_{n-1}^i, V_{n-1}^j aus dem Mittelpunkt von T_n, so haben wir eine Anzahl von Simplexe von n-Dimensionen welche mit der Simplexen t_n^i eine Einteilung des gegebenen Simplex T_n in kleineren geben. Wenn der Durchmesser von T_n d ist der von jedes oben gewonnen en kleinen Simplexe nicht grösser

als $\frac{d'}{2}$.

Fahren wir diese verfahren fort, so können wir das gegebene Simplex in kleineren von Durchmessern minder als eine gegebenen positeven Zahl einteilen.

Wir betonen, dass diese Verfahren auf zwei oder mehr mit gemeinsamen Seiten benachbarten Simplexen T_n gleichzeitig ausgeführt werden können, da diese Einteilung durch die der Seitensimplexen geleitet ist.

Sur la Classe Quasi-analytique de Fonctions de Deux Variables (IV)

Par **Sikazô Kodama**

(Reçu en Octobre 7, 1940)

Nous traitons maintenant la quasi-analyticité de fonctions doublement périodiques exprimées par les séries doubles de Fourier, et celle faisant usage des transformées de Fourier avec les intégrales considérées au sens de Cesàro dans la classe L^{*p} des fonctions de deux variables.

I. La Quasi-analyticité des séries doubles de Fourier

§ 1. Un théorème préliminaire

1. Désignons, comme plus haut, par C_M une classe de fonctions, de deux variables x et y, indéfiniment dérivables dans le domaine $\begin{pmatrix} 0 \leqq x \leqq 2\pi \\ 0 \leqq y \leqq 2\pi \end{pmatrix}$, et satisfaisant aux conditions suivantes :

$$(1.1) \qquad |f^{(m+n)}(x,y)| \leqq k^{m+n} M_{m,n} \qquad \begin{pmatrix} 0 \leqq x \leqq 2\pi\,;\ m=0,1,2,\cdots \\ 0 \leqq y \leqq 2\pi\,;\ n=0,1,2,\cdots \end{pmatrix},$$

où $\{M_{m,n}\}\begin{pmatrix} m=0,1,2,\cdots \\ n=0,1,2,\cdots \end{pmatrix}$ est une suite double donnée à termes positifs et k est, comme on le sait, une constante positive dépendant seulement du choix de la fonction $f(x,y)$ dans cette classe. Posons, ainsi qu'on a souvent fait, $\beta_{m,n}=\sqrt[m+n]{M_{m,n}}\begin{pmatrix} m=0,1,2,\cdots \\ n=0,1,2,\cdots \end{pmatrix}$, et désignons aussi par $\{\beta^*_{m,n}\}\begin{pmatrix} m=0,1,2,\cdots \\ n=0,1,2,\cdots \end{pmatrix}$ la minorante de M. Faber de la suite double donnée $\{\beta_{m,n}\}\begin{pmatrix} m=0,1,2,\cdots \\ n=0,1,2,\cdots \end{pmatrix}$. Alors on sait que la condition[1] nécessaire et suffisante pour la quasi-analyticité de la classe C_M est donnée par la divergence de la série double $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\beta^*_{m,n}}$, c'est-à-dire celle de l'intégrale double

$$\int_1^\infty\int_1^\infty \log T(R,r)\frac{dR}{R^2}\frac{dr}{r^2} \qquad \left[\text{où } T(R,r)=\max_{m,n\geq 0}\frac{R^m r^n}{\beta^{m+n}_{m,n}}\right],$$

R et r étant des valeurs absolues de x et y.

1. S. Kodama : Sur la Classe Quasi-analytique de Fonctions de Deux Variables (I) (Memoirs of the College of Science Kyoto Imperial University, Series A, Vol. XXII, 1939, p. 309).

Il est à remarquer ici la proposition générale suivante :

Si une série double à termes positifs $\sum\sum a_{m,n}$ *diverge et si l'on pose* $a^*_{m,n}=\min_{m'+n'\geq m+n} a_{m',n'}$, *alors* $\varlimsup_{m+n\to\infty} a^*_{m,n}=\varlimsup_{p\to\infty} A_p$, *où* A_p *désigne* $\min_{m+n=p} a^*_{m,n}$.

On peut démontrer cette proposition de la manière suivante. Posons, tout d'abord, $\lambda=\varlimsup_{m+n\to\infty} a^*_{m,n}$. Alors on peut en déduire que $\lambda=\varlimsup_{p\to\infty} A_p$. En effet, désignant par δ un nombre positif quelconque aussi petit que l'on veut, il n'existe qu'un nombre fini de $a^*_{m,n}$ plus petit que $\lambda-\delta$, tandis qu'il y a une infinité de $a^*_{m,n}$ plus petits que $\lambda+\delta$. Il existe donc une infinité de $a^*_{m,n}$ dans l'intervalle $(\lambda-\delta, \lambda+\delta)$, désigné par Λ_δ, dont les éléments peuvent s'écrire sous la forme :

$$a^*_{m_1,n_1},\ a^*_{m_2,n_2},\ \ldots\ldots,\ a^*_{m_j,n_j},\ \ldots\ldots ;$$

d'où, en posant $p_j=m_j+n_j$, on tire $A_{p_j}\leqslant a^*_{m_j,n_j}$, et par suite on voit que $A_{p_j}\leqslant\lambda+\delta$. Donc les A_{p_j}, excépté son fini, sont contenus dans Λ_δ, ce qui est satisfait pour toute valeur de δ. Il en résulte que λ est un élément limite des A_{p_j} (et donc des A_p). Or il n'y a qu'un nombre fini de A_p plus petit que $\lambda-\delta$. Donc on a $\lambda=\varlimsup_{p\to\infty} A_p$.

Réciproquement, si l'on pose $\lambda=\varlimsup_{p\to\infty} A_p$, on peut en déduire que $\lambda=\varlimsup_{m+n\to\infty} a^*_{m,n}$. Étant, en effet, donné un nombre positif δ quelconque aussi petit que l'on veut, comme on a $\lambda=\varlimsup_{p\to\infty} A_p$ par hypothèse, il n'existe qu'un nombre fini de A_p plus petit que $\lambda-\delta$. On désigne ces valeurs par $A_{p_1}, A_{p_2}, \ldots, A_{p_i}$. Et les autres valeurs de $a^*_{m,n}$ sont au moins $\lambda-\delta$. On voit, d'après la définition même de A_p, que les seuls nombres de $a^*_{m,n}$ pour lesquels la somme des indices m et n prend les valeurs $p_1, p_2, \ldots, p_i$ de la suite $\{a^*_{m,n}\}$ inférieurs à $\lambda-\delta$ font partie de cette suite. Or, cette dernière suite est construite par un nombre fini de termes. D'où l'on voit qu'il n'y existe qu'un nombre fini de $a^*_{m,n}$ plus petit que $\lambda-\delta$. Cette propriété a lieu quelque petit que soit le nombre positif δ. Si donc on considère que A_p font partie de la suite double $\{a^*_{m,n}\}$, alors on voit que λ est un élément limite de cette suite, et par conséquent que $\lambda=\varlimsup_{m+n\to\infty} a^*_{m,n}$.

2. Démontrons maintenant le théorème suivant :

Si, en particulier, $f(x,y)$ *est une fonction doublement périodique, de périodes* 2π, 2π *par rapport à* x *et à* y, *et pouvant s'écrire sous la forme* :

$$(2.1)\qquad f(x,y)=\frac{1}{4}a_{0,0}+\frac{1}{2}\sum_{m=1}^{\infty}(a_{m,0}\cos mx+b_{m,0}\sin mx)$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}(a_{0,n}\cos ny+c_{0,n}\sin ny)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{m,n}\cos mx\cos ny$$
$$+b_{m,n}\sin mx\cos ny+c_{m,n}\cos mx\sin ny+d_{m,n}\sin mx\sin ny),$$

alors on peut toujours trouver une fonction $\phi(s,t)$ *jouissant des propriétés suivantes*: 1° *Elle est positive, continue et dérivable dans le domaine* $\begin{pmatrix}s>0\\t>0\end{pmatrix}$; 2° *Pour* $t=\nu s$, ν *étant un paramètre positif, le produit* $s\frac{d}{ds}\phi(s,\nu s)$ *tend vers l'infini avec* s; 3° *L'intégrale double*

$$(2.2)\qquad \int_1^{\infty}\int_1^{\infty}\phi(s,t)\frac{ds}{s^2}\frac{dt}{t^2}$$

diverge; 4° *Et de plus, les inégalités suivantes sont remplies*:

$$(2.3)\qquad |a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|<e^{-\phi(m,n)}\qquad \begin{pmatrix}m=0,1,2,\dots\\n=0,1,2,\dots\end{pmatrix}.$$

On sait, en effet, en vertu des propriétés bien connues de la série double de Fourier, que

$$\begin{matrix}a_{m,n}\\b_{m,n}\\c_{m,n}\\d_{m,n}\end{matrix}=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}f(x,y)\begin{matrix}\cos\\\sin\\\cos\\\sin\end{matrix}mx\begin{matrix}\cos\\\cos\\\sin\\\sin\end{matrix}ny\,dxdy;$$

d'où l'on tire aisément

$$(2.4)\qquad |a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|\leqslant\frac{k^{p+q}M_{p,q}}{\pi^2m^pn^q}\int_0^{2\pi}\int_0^{2\pi}dxdy$$
$$=\frac{4}{T\left(\frac{m}{k},\frac{n}{k}\right)},$$

les entiers positifs p et q étant choisis tels qu'on ait

$$T\left(\frac{m}{k},\frac{n}{k}\right)=\frac{m^pn^q}{(k\beta_{p,q})^{p+q}}.$$

Supposons, tout d'abord, que $\varlimsup_{p+q\to\infty}\beta_{p,q}^{*}<\infty$. Alors on a $T(R,r)=\infty$ pour R,r suffisamment grands; et par suite, en prenant des entiers m et n plus grands que k, et en prenant des entiers positifs p et q assez grands, la quantité $\frac{m^pn^q}{(k\beta_{p,q})^{p+q}}$ devient suffisamment grande, ce qui donne $T\left(\frac{m}{k},\frac{n}{k}\right)=\infty$; et l'on a donc, à partir de certains rangs des entiers positifs m, n,

$$|a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|=0,$$

c'est-à-dire, on a, pour telles valeurs de m et n,

$$a_{m,n},\ b_{m,n},\ c_{m,n},\ d_{m,n}=0.$$

Il n'y a donc qu'un nombre fini de termes non nuls de la série double de Fourier de la fonction $f(x, y)$. Nous pouvons donc, dans ce cas, trouver facilement une fonction $\phi(s, t)$ jouissant des propriétés 1°, 2°, 3° et 4° écrites plus haut. Il est aisé de voir, par exemple, que la fonction $\phi(s, t)=st \log s \log_2 t$ satisfait certainement aux quatre conditions.

Supposons, ensuite, que $\lim_{p+q\to\infty} \beta^*_{p,q}=\infty$. L'intégrale double

$$\int_1^\infty\int_1^\infty \log T(R\ r)\frac{dR}{R^2}\frac{dr}{r^2}$$

diverge par hypothèse. On sait, en plus, que la somme $N(R, r)=\lambda(R, r)+\mu(R, r)$ dans l'expression $T(R, r)=\dfrac{R^\lambda r^\mu}{\beta^{\lambda+\mu}_{\lambda,\mu}}$ donnée par la formule

$$\log T(R, r)=\int_0^1 N(R\xi, r\xi)\frac{d\xi}{\xi}$$

est discontinue sur certaines lignes, croissante, et tend vers l'infini avec chaque variable R, r. Construisons donc une fonction $\psi(R, r)$ continue, croissante, tendant vers l'infini avec R, r, et coïncidant avec la fonction $N(R, r)$ sauf au voisinage de discontinuités, et de plus vérifiant la relation asymptotique suivante :

$$\bar{\psi}(R, r)=\int_0^1 \psi(R\xi, r\xi)\frac{d\xi}{\xi}\sim\int_0^1 N(R\xi, r\xi)\frac{d\xi}{\xi}=\log T(R, r).$$

On peut alors trouver une constante positive finie l telle que, pour $R, r\geqslant 1$, on a

$$l\ \bar{\psi}(R, r)<\log T(R, r).$$

En posant après cela pour $R, r>1$

$$\phi(kR, kr)=l\,\bar{\psi}(R, r)-C-\log 4,$$

où C est une constante positive fixe mais quelconque, on a alors

$$\phi(m, n)=l\,\bar{\psi}\left(\frac{m}{k},\ \frac{n}{k}\right)-C-\log 4\ ;$$

il en résulte que

$$(2.5)\qquad \begin{aligned} e^{-\phi(m,n)} &= 4e^{C-l\bar{\psi}\left(\frac{m}{k},\ \frac{n}{k}\right)} > 4e^{C-\log T\left(\frac{m}{k},\ \frac{n}{k}\right)} \\ &= \frac{4e^C}{T\left(\dfrac{m}{k},\ \dfrac{n}{k}\right)} \geqslant e^C \begin{vmatrix} a_{m,n} & \\ b_{m,n} & \\ c_{m,n} & \\ d_{m,n} & \end{vmatrix} > \begin{vmatrix} a_{m,n} & \\ b_{m,n} & \\ c_{m,n} & \\ d_{m,n} & \end{vmatrix}. \end{aligned}$$

Comme, d'ailleurs, on a, pour un paramètre postif ν tel que $r=\nu R$,

$$k\frac{d}{dR}\phi(kR, \nu kR)=\frac{l}{R}N(R, \nu R),$$

on voit que la fonction

$$(2.6) \qquad s\frac{d}{ds}\phi(s,\nu s)=lN\left(\frac{s}{k},\frac{\nu s}{k}\right)$$

tend vers l'infini avec s.

Par conséquent on voit : 1° que cette fonction $\phi(s,t)$ est évidemment positive, continue et dérivable dans le domaine $\begin{pmatrix}s>0\\t>0\end{pmatrix}$; 2° que la fonction $s\frac{d}{ds}\phi(s,\nu s)$ tend vers l'infini avec s pour toute valeur d'un paramètre positif ν ; 3° que l'intégrale double $\int_1^\infty\int_1^\infty \phi(s,t)\frac{ds}{s^2}\frac{dt}{t^2}$ diverge, car on sait que l'intégrale double $\int_1^\infty\int_1^\infty \log T(R,r)\frac{dR}{R^2}\frac{dr}{r^2}$ diverge par hypothèse, et de plus, que $\phi(s,t)\sim l\log T\left(\frac{s}{k},\frac{t}{k}\right)$; 4° et finalement que

$$|a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|<e^{-q(m,n)} \qquad \begin{pmatrix}m=0,1,2,\cdots\\n=0,1,2,\cdots\end{pmatrix}.$$

On voit donc que cette fonction $\phi(s,t)$ est exactement la fonction juissant des quatre propriétés écrites plus haut, ce qui démontre notre théorème.

§ 2. L'extension d'un théorème de M. Mandelbrojt

3. En remarquant le théorème donné dans le dernier numéro, nous pouvons maintenant traiter avec M. Mandelbrojt un problème remarquable, qui peut s'énoncer de la manière suivante :

Soit $f(x,y)$ une fonction indéfiniment dérivable, définie par la série double (2.1), *et satisfaisant aux conditions suivantes :*

$$f^{(p+q)}(0,0)=0 \qquad \begin{pmatrix}p=0,1,2,\cdots\\q=0,1,2,\cdots\end{pmatrix},$$

$$|a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|<\lambda_{m,n} \qquad \begin{pmatrix}m=0,1,2,\cdots\\n=0,1,2,\cdots\end{pmatrix}.$$

Quelles sont alors les conditions nécessaires et suffisantes qu'il faut imposer à la suite double $\{\lambda_{m,n}\}$ $\begin{pmatrix}m=0,1,2,\cdots\\n=0,1,2,\cdots\end{pmatrix}$, pour que la fonction $f(x,y)$ soit identiquement nulle.

Si $\lambda_{m,n}\begin{pmatrix}m=0,1,2,\cdots\\n=0,1,2,\cdots\end{pmatrix}$ peut être mise sous la forme

$$(3.1) \qquad \lambda_{m,n}=e^{-\phi(m,n)}\begin{pmatrix}m=0,1,2,\cdots\\n=0,1,2,\cdots\end{pmatrix},$$

où $\phi(s,t)$ est une fonction satisfaisant aux conditions suivantes : 1° Elle est positive, continue et dérivable dans le domaine $\begin{pmatrix}s>0\\t>0\end{pmatrix}$; 2° La fonc-

tion $s\frac{d}{ds}\phi(s, \nu s)$ tend vers l'infini en croissant avec s pour toute valeur positive d'un paramètre ν ; on peut alors énoncer avec M. Mandelbrojt[1] deux théorèmes répondant à ce problème de la manière suivante :

I. *Si l'intégrale double*

$$(3.2) \qquad \int_1^\infty\int_1^\infty \phi(s, t)\frac{ds}{s^2}\frac{dt}{t^2}$$

converge, on peut alors construire une fonction $f(x, y)$ non identiquement nulle, indéfiniment dérivable, paire-paire, doublement périodique, de périodes 2π, 2π, satisfaisant aux conditions suivantes :

$$f^{(p+q)}(0, 0)=0 \qquad \begin{pmatrix} p=0, 1, 2, \cdots \\ q=0, 1, 2, \cdots \end{pmatrix},$$

et de plus, telle qu'en posant

$$(3.3) \qquad f(x, y)=\frac{1}{4}a_{0,0}+\frac{1}{2}\sum_{m=1}^{\infty}a_{m,0}\cos mx+\frac{1}{2}\sum_{n=1}^{\infty}a_{0,n}\cos ny + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\cos mx\cos ny,$$

on ait

$$(3.4) \qquad |a_{m,n}|<e^{-\phi(m,n)} \qquad \begin{pmatrix} m=0, 1, 2, \cdots \\ n=0, 1, 2, \cdots \end{pmatrix}.$$

II. *Si, par contre, l'intégrale double* (3.2) *diverge, et si la fonction $f(x, y)$ vérifiant les relations :*

$$f^{(p+q)}(0, 0)=0 \qquad \begin{pmatrix} p=0, 1, 2, \cdots \\ q=0, 1, 2, \cdots \end{pmatrix}$$

est développable en série double de Fourier (2.1) *dont les coefficients satisfont aux conditions suivantes :*

$$(3.5) \qquad |a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|<e^{-\psi(m,n)} \qquad \begin{pmatrix} m=0, 1, 2, \cdots \\ n=0, 1, 2, \cdots \end{pmatrix};$$

alors la fonction $f(x, y)$ est identiquement nulle.

Si l'on remarque, d'une part, que la somme $N(R, r)=\lambda(R, r)+\mu(R, r)$ dans l'expression $\mathfrak{M}(R, r)=\max_{m,n\geq 0} C_{m,n}R^m r^n=C_{\lambda,\mu}R^\lambda r^\mu$ est non décroissante et tend vers l'infini avec R, r ; et d'autre part, que la fonction $s\frac{d}{ds}\phi(s, \nu s)$ tend vers l'infini en croissant avec s pour toute valeur positive d'un paramètre ν ; on peut alors désigner $N(R, \nu R)$ la partie entière de $R\frac{d}{dR}\phi(R, \nu R)$, telle qu'on ait

$$R\phi'_R(R, \nu R)=N(R, \nu R)+\phi_1(R, \nu R),$$

1. S. Mandelbrojt : Quasi-analycité des séries de Fourier (Annali della R. Scuola Normale Superiore di Pisa, Serie II vol. IV (1935) pp. 225–229).

où $0 \leqslant \phi_1(R, \nu R) < 1$. On en déduit que

$$(3.6) \qquad \phi(1, \nu) + \int_1^R \frac{N(s, \nu s)}{s} ds \leqslant \phi(R, \nu R) < \phi(1, \nu) + \log R + \int_1^R \frac{N(s, \nu s)}{s} ds ;$$

d'où l'on tire

$$(3.7) \qquad \phi(1, \nu) + \log \mathfrak{M}(R, r) \leqslant \phi(R, r) < \phi(1, \nu) + \log\{R\mathfrak{M}(R, r)\},$$

qui est la seule condition exigée pour la fonction $\phi(R, r)$.

Nous allons maintenant démontrer le théorème I. Posons, pour cela,

$$T(R, r) = \max_{m, n \geq 1} e^{\phi(1, \nu)} C_{m-1, n} R^m r^n = e^{\phi(1, \nu)} R\, \mathfrak{M}(R, r) ;$$

on a alors, en vertu des inégalités (3.7),

$$(3.8) \qquad e^{\phi(R, r)} < T(R, r).$$

On peut aussi, en vertu de ces mêmes inégalités et de l'hypothèse que l'intégrale double (3.2) converge, affirmer que l'intégrale double

$$\int_1^\infty \int_1^\infty \log T(R, r) \frac{dR}{R^2} \frac{dr}{r^2} = \phi(1, \nu) \int_1^\infty \int_1^\infty \frac{dR}{R^2} \frac{dr}{r^2} + \int_1^\infty \int_1^\infty \log R \frac{dR}{R^2} \frac{dr}{r^2} + \int_1^\infty \int_1^\infty \log \mathfrak{M}(R, r) \frac{dR}{R^2} \frac{dr}{r^2}$$

converge aussi. Posons donc $M_{m, n} = e^{-\phi(1, \nu)} / C_{m-1, n}$. On peut alors construire, grâce au théorème du n° 18 de mon mémoire (I), une fonction $f(x, y)$, non identiquement nulle, indéfiniment dérivable dans le domaine $\begin{pmatrix} 0 \leqslant x \leqslant 2\pi \\ 0 \leqslant y \leqslant 2\pi \end{pmatrix}$, et satisfaisant aux conditions suivantes :

$$(3.9) \qquad f^{(p+q)}(0, 0) = f^{(p+q)}(0, 2\pi) = f^{(p+q)}(2\pi, 0) = f^{(p+q)}(2\pi, 2\pi) = 0 \qquad \begin{pmatrix} p = 0, 1, 2, \ldots \\ q = 0, 1, 2, \ldots \end{pmatrix},$$

$$(3.10) \qquad |f^{(p+q)}(x, y)| < \frac{1}{4} M_{p, q} \qquad \begin{pmatrix} 0 \leqslant x \leqslant 2\pi ; \ p = 0, 1, 2, \ldots \\ 0 \leqslant y \leqslant 2\pi ; \ q = 0, 1, 2, \ldots \end{pmatrix},$$

$$f(x, y) = f(2\pi - x, 2\pi - y) \qquad \begin{pmatrix} 0 \leqslant x \leqslant 2\pi \\ 0 \leqslant y \leqslant 2\pi \end{pmatrix}.$$

Lorsque cette fonction $f(x, y)$ est prolongée périodiquement avec les périodes 2π, 2π, elle est paire-paire et peut s'écrire sous la forme :

$$f(x, y) = \frac{a_{0, 0}}{4} + \frac{1}{2} \sum_{m=1}^{\infty} a_{m, 0} \cos mx + \frac{1}{2} \sum_{n=1}^{\infty} a_{0, n} \cos ny + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m, n} \cos mx \cos ny ;$$

et par suite, en tenant compte des conditions (3.9) on a, d'après intégrations par parties,

$$a_{m,n}=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\cos mx\cos ny\,dx\,dy$$

$$=\frac{\pm 1}{\pi^2 m^p n^q}\int_0^{2\pi}\int_0^{2\pi} f^{(p+q)}(x,y)\begin{matrix}\cos\\ \sin\end{matrix} mx\begin{matrix}\cos\\ \sin\end{matrix} ny\,dx\,dy\,;$$

et, grâce à l'inégalité (3.10), on peut en tirer pour tous $m, n\geqslant 0$,

$$|a_{m,n}|<\frac{M_{p,q}}{m^p n^q}\qquad \begin{pmatrix} p=0,1,2,\dots\\ q=0,1,2,\dots\end{pmatrix}.$$

Il en résulte évidemment que

$$|a_{m,n}|\leqslant\frac{1}{\max\limits_{p,q\geq 1}\dfrac{m^p n^q}{M_{p,q}}}=\frac{1}{T(m,n)}<e^{-\varphi(m,n)}$$

$$\begin{pmatrix} m=0,1,2,\dots\\ n=0,1,2,\dots\end{pmatrix},$$

ce qui prouve notre théorème I.

4. Passons ensuite à la démonstration de notre théorème II. D'après les inégalités (3.5) et (3.7), on a

$$|a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|\leqslant e^{-\varphi(1,\nu)-\log\mathfrak{M}(m,n)}$$

$$\leqslant\frac{l}{\mathfrak{M}(m,n)}\,;$$

d'où il résulte, en vertu de la définition même de $\mathfrak{M}(R,r)$, que

$$|a_{m,n}|,\ |b_{m,n}|,\ |c_{m,n}|,\ |d_{m,n}|<\frac{l}{C_{p,q}m^p n^q}\qquad (4.1)$$

$$\begin{pmatrix} m=0,1,2,\dots\\ n=0,1,2,\dots\end{pmatrix},$$

quels que soient les entiers positifs p et q.

Comme la fonction $f(x,y)$ est indéfiniment dérivable par hypothèse, et comme sa série double de Fourier est uniformément convergente dans son domaine d'existence et dérivable terme-à-terme, on a, quels que soient les entiers positifs k_1 et k_2,

$$f^{(k_1+0)}(x,y)=\pm\sum_{m=1}^{\infty}\left(a_{m,0}\begin{matrix}\cos\\ \sin\end{matrix} mx+b_{m,0}\begin{matrix}\sin\\ \cos\end{matrix} mx\right)m^{k_1}$$

$$\pm\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(a_{m,n}\begin{matrix}\cos\\ \sin\end{matrix} mx\cos ny+b_{m,n}\begin{matrix}\sin\\ \cos\end{matrix} mx\cos ny$$

$$+c_{m,n}\begin{matrix}\cos\\ \sin\end{matrix} mx\sin ny+d_{m,n}\begin{matrix}\sin\\ \cos\end{matrix} mx\sin ny\Big)\Big)m^{k_1},$$

$$f^{(0+k_2)}(x,y)=\pm\sum_{n=1}^{\infty}\left(a_{0,n}\begin{matrix}\cos\\ \sin\end{matrix} ny+c_{0,n}\begin{matrix}\sin\\ \cos\end{matrix} ny\right)n^{k_2}$$

$$\pm\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(a_{m,n}\cos mx\begin{matrix}\cos\\ \sin\end{matrix}ny+b_{m,n}\sin mx\begin{matrix}\cos\\ \sin\end{matrix}ny$$

$$+c_{m,n}\cos mx\begin{matrix}\sin\\ \cos\end{matrix}ny+d_{m,n}\sin mx\begin{matrix}\sin\\ \cos\end{matrix}ny\Big)n^{k_2},$$

$$f^{(k_1+k_2)}(x,y)=\pm\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(a_{m,n}\begin{matrix}\cos\\ \sin\end{matrix}mx\begin{matrix}\cos\\ \sin\end{matrix}ny+b_{m,n}\begin{matrix}\sin\\ \cos\end{matrix}mx\begin{matrix}\cos\\ \sin\end{matrix}ny$$

$$+c_{m,n}\begin{matrix}\cos\\ \sin\end{matrix}mx\begin{matrix}\sin\\ \cos\end{matrix}ny+d_{m,n}\begin{matrix}\sin\\ \cos\end{matrix}mx\begin{matrix}\sin\\ \cos\end{matrix}ny\Big)m^{k_1}n^{k_2};$$

d'où l'on tire donc, pour toutes valeurs positives de deux entiers k_1 et k_2,

$$|f^{(k_1+0)}(x,y)|\leqslant\sum_{m=1}^{\infty}(|a_{m,0}|+|b_{m,0}|)m^{k_1}$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(|a_{m,n}|+|b_{m,n}|+|c_{m,n}|+|d_{m,n}|)m^{k_1},$$

$$|f^{(0+k_2)}(x,y)|\leqslant\sum_{n=1}^{\infty}(|a_{0,n}|+|c_{0,n}|)n^{k_2}$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(|a_{m,n}|+|b_{m,n}|+|c_{m,n}|+|d_{m,n}|)n^{k_2},$$

$$|f^{(k_1+k_2)}(x,y)|\leqslant\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(|a_{m,n}|+|b_{m,n}|+|c_{m,n}|+|d_{m,n}|)m^{k_1}n^{k_2};$$

et par conséquent, en vertu des inégalités (4.1) où l'on pose $p=k_1+2$, $q=k_2+2$, on a

$$|f^{(k_1+0)}(x,y)|<2l\sum_{m=1}^{\infty}\frac{m^{k_1}}{C_{k_1+2,0}m^{k_1+2}}$$

$$+4l\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{k_1}}{C_{k_1+2,0}m^{k_1+2}n^2}=\frac{N}{C_{k_1+2,0}},$$

$$|f^{(0+k_2)}(x,y)|<2l\sum_{n=1}^{\infty}\frac{n^{k_2}}{C_{0,k_2+2}n^{k_2+2}}$$

$$+4l\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^{k_2}}{C_{0,k_2+2}m^2n^{k_2+2}}=\frac{N}{C_{0,k_2+2}},$$

$$|f^{(k_1+k_2)}(x,y)|<4l\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{k_1}n^{k_2}}{C_{k_1+2,k_2+2}m^{k_1+2}n^{k_2+2}}$$

$$<\frac{N}{C_{k_1+2,k_2+2}},$$

N étant une certaine constante positive finie.

Posons donc

$$\theta(x,y)=\int_0^x d\sigma\int_0^y d\tau\int_0^\sigma\int_0^\tau f(\xi,\eta)d\xi\,d\eta.$$

Il est alors aisé de vérifier les conditions suivantes :

$$|\theta^{(p+q)}(x,y)|<NM_{p,q}\qquad\begin{pmatrix}0\leqslant x\leqslant 2\pi\,;\,p=0,1,2,\ldots,\\ 0\leqslant y\leqslant 2\pi\,;\,q=0,1,2,\ldots,\end{pmatrix},$$

$$\theta^{(p+q)}(0, 0)=0 \qquad \begin{pmatrix} p=0, 1, 2, \cdots \\ q=0, 1, 2, \cdots \end{pmatrix},$$

où l'on désigne $M_{p,q}=1/C_{p,q}$ pour $p, q \geqslant 3$. Or, l'intégrale double (3.2) diverge par hypothèse ; d'où il résulte que l'intégrale double

$$\int_1^\infty\int_1^\infty \log \mathfrak{M}(R, r)\frac{dR}{R^2}\frac{dr}{r^2}$$

diverge, et par suite l'intégrale double

$$\int_1^\infty\int_1^\infty \log T(R, r)\frac{dR}{R^2}\frac{dr}{r^2}$$

diverge aussi. On voit donc que, en vertu du théorème du n° 18 de mon mémoire (I), cette fonction $\theta(x, y)$ est identiquement nulle dans le domaine $\begin{pmatrix} 0\leqslant x\leqslant 2\pi \\ 0\leqslant y\leqslant 2\pi \end{pmatrix}$, et par conséquent que, par son prolongement, cette fonction est identiquement nulle dans tous domaines. On en conclut que la fonction $f(x, y)=\dfrac{\partial^4}{\partial x^2\partial y^2}\theta(x, y)$ l'est aussi. Notre théorème II est aussi démontré.

On voit donc avec M. Mandelbrojt que si l'on peut poser

$$\lambda_{m,n}=e^{-\phi(m,n)} \qquad \begin{pmatrix} m=0, 1, 2, \cdots \\ n=0, 1, 2, \cdots \end{pmatrix},$$

où la fonction $\phi(s, t)$ est positive, continue et dérivable, telle que la fonction $s\dfrac{d}{ds}\phi(s, \nu s)$ tend vers l'infini en croissant avec s pour toute valeur positive d'un paramètre ν, alors la condition nécessaire et suffisante pour que la fonction $f(x, y)$ donnée par (2.1) soit identiquement nulle est que l'intégrale double (3.2) diverge.

II. La transformée de Fourier

§ 1. Celle dont les intégrales sont considérées au sens de Cesàro

5. Afin d'obtenir des résultats analogues quand on fait usage des transformées de Fourier, nous désignons maintenant brièvement la notion de la transformée de Fourier, nécessaire pour l'usage futur, d'une fonction de deux variables. Considérons,[1] pour cela, deux fonctions liées par les relations suivantes :

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1. A. C. Berry : *Necessary and sufficient conditions in the theory of Fourier transforms* (Annales of Math. (2), 32 (1931), pp. 830-838). S. Bochner : *Ein Konvergenzsatz für mehrvariablige Fouriersche Integrale* (Math. Zeitschrift, 34 (1932) SS. 440-447). A. C. Offord : *On Fourier transforms* I, II, III (Proc. of Lond. Math. Soc. (2), 38 (1935) pp. 197-216 ; Proc. of Lond. Math. Soc. (2), 40 (1935) pp. 281-289 ; Trans. Amer. Math. Soc., 38 (1935), pp. 250-266).

$$(5.1)\qquad \begin{aligned} F(x,y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)e^{-i(xu+yv)}\,du\,dv,\\ f(x,y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i(xu+yv)}\,du\,dv. \end{aligned}$$

Ces integrales sont ici considérées au sens de Cesàro, c'est-à-dire, si d'une manière générale $\lim_{U,V\to\infty}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)^{\alpha}\left(1-\frac{|v|}{V}\right)^{\beta}\phi(u,v)\,du\,dv$ tend vers une quantité finie, on dit alors que l'intégrale double $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u,v)\,du\,dv$ est *sommable*$-(C,\alpha,\beta)$ où l'on suppose que α et $\beta>0$. En posant

$$(5.2)\qquad F(x,y;\,U,V)=\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)\times f(u,v)e^{-i(xu+yv)}\,du\,dv,$$

nous supposons, pour toutes valeurs de U et V, que

$$F(x,y;\,U,V)\in L^{p}\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}\qquad[1\leqslant p\leqslant\infty].$$

S'il existe, dans ce cas, une fonction $F(x,y)$, telle qu'on ait

$$\lim_{U,V\to\infty}\int_{-U}^{U}\int_{-V}^{V}|F(x,y)-F(x,y;\,U,V)|^{p}\,dx\,dy=0\qquad[1\leqslant p<\infty],$$

et

$$\lim_{U,V\to\infty}\text{borne ess. sup. }|F(x,y)-F(x,y;\,U,V)|=0\qquad[p=\infty],$$

alors on dit que $F(x,y)$ est *la transformée de Fourier* dans L^{p} de la fonction $f(x,y)$.

Lorsqu'une fonction $f(x,y)$ intégrable-L dans tout domaine fini satisfait à la condition suivante :

$$(5.3)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y;\,U,V)|^{p}dxdy\leqslant M^{p}\qquad[1<p<\infty]$$

où M est une quantité finie indépendante de U et de V, on dit que la fonction $f(x,y)$ appartient à *la classe* $\mathfrak{H}^{p}[1<p<\infty]$. Lorsqu'on a

$$(5.4)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y;\,U,V)|\,dxdy\leqslant M,$$

et, en plus, que, pour $\varepsilon(>0)$ assez petite donnée à l'avance, on a

$$(5.5)\qquad \iint_{\mathfrak{A}}|F(x,y;\,U,V)|\,dxdy\leqslant\varepsilon\qquad[\text{où } m(\mathfrak{A})<\delta(\varepsilon)],$$

on dit que la fonction $f(x,y)$ appartient à *la classe* $\mathfrak{H}^{1}$. Lorsqu'on a, pour toutes valeurs de x,y et de U,V.

$$(5.6)\qquad |F(x,y;\,U,V)|\leqslant M,$$

on dit que la fonction $f(x,y)$ appartient à *la classe* $\mathfrak{H}^{\infty}$.

6. Prenons maintenant une fonction $f(x, y)$ dans la classe $\mathfrak{H}^p$ $[1<p<\infty]$, et considérons la fonction

$$\phi(x, y;\ U, V)=\int_a^x\int_b^y F(u, v;\ U, V)dudv$$

dans un domaine quelconque $\begin{pmatrix} a\leqq x\leqq A \\ b\leqq y\leqq B\end{pmatrix}$. Comme il est aisé de voir que

$$\phi(a, b;\ U, V)=0, \qquad |\phi(x, y;\ U, V)|\leqq NM,$$

où N est une constante positive finie, on voit que la famille des fonctions ϕ est également continue. On peut donc trouver une suite de fonctions $\{\phi(x, y;\ U_j, V_j)\}(j=1, 2, 3, \ldots)$ convergeant uniformément vers une fonction $\Phi(x, y;\ U, V)$ de cette famille. Considérons, ensuite, une suite de fonctions $\{F(x, y;\ U_j, V_j)\}$ correspondante à la suite trouvée $\{\phi(x, y,\ U_j, V_j)\}$. En divisant le domaine $\begin{pmatrix} a\leqq x\leqq A \\ b\leqq y\leqq B\end{pmatrix}$ en mn parties par m parallèles à l'axe des x et par n parallèles à l'axe des y passant par les points $x_\lambda(\lambda=1, 2, \ldots, m-1)$ et $y_\mu(\mu=1, 2, \ldots, n-1)$ supposant qu'on ait $\begin{matrix} x_0=a, & x_m=A \\ y_0=b, & y_n=B\end{matrix}$; posons, pour abréger,

$$\phi_j\begin{pmatrix} x_\lambda, & x_{\lambda+1} \\ y_\mu, & y_{\mu+1}\end{pmatrix}=\phi(x_{\lambda+1}, y_{\mu+1};\ U_j, V_j)-\phi(x_{\lambda+1}, y_\mu;\ U_j, V_j)$$
$$-\phi(x_\lambda, y_{\mu+1};\ U_j, V_j)+\phi(x_\lambda, y_\mu;\ U_j, V_j);$$

alors on a

$$\sum_{\lambda=0}^{m-1}\sum_{\mu=0}^{n-1}\frac{\left|\phi_j\begin{pmatrix} x_\lambda, & x_{\lambda+1} \\ y_\mu, & y_{\mu+1}\end{pmatrix}\right|^p}{\{(x_{\lambda+1}-x_\lambda)(y_{\mu+1}-y_\mu)\}^{p-1}}$$
$$\leqq\int_a^A\int_b^B|F(x, y;\ U_j, V_j)|^p dxdy\leqq M^p,$$

qui est la condition nécessaire et suffisante pour que la fonction $\Phi(x, y;\ U, V)$ soit une intégrale indéfinie $\int_a^x\int_b^y F(u, v)dudv$ d'une fonction $F(x, y)$ de la classe L^p. On peut donc trouver une fonction $F(x, y)$ correspondante à la suite $\{F(x, y;\ U_j, V_j)\}(j=1, 2, 3, \ldots)$, telle qu'on ait

$$\lim_{j\to\infty}\int_a^x\int_b^y F(u, v;\ U_j, V_j)|dudv=\int_a^x\int_b^y F(u, v)dudv,$$

où $F(u, v)\in L^p$. Cette suite $\{F(x, y;\ U_j, V_j)\}$ converge donc faiblement, avec un exposant p, vers cette fonction $F(x, y)$ même. Par un léger changement, on peut de même affirmer cette propriété dans les cas où $p=1$ où ∞. Nous avous donc le théorème suivant:

Si $f(x, y)\in\mathfrak{H}^p[1\leqq p\leqq\infty]$, *alors on peut toujours trouver une suite accouplée* $\{(U_j, V_j)\}(j=1, 2, 3, \ldots)$, *telle qu'on ait*

(6.1) $\quad F(x, y;\ U_j, V_j)\underset{(p)}{\rightarrow}F(x, y)\in L^p$, *faiblement.*

7. D'où l'on peut tirer sans difficulté le corollaire suivant :

Soit $f(x,y)\in\mathfrak{H}^p$ $[1\leqslant p\leqslant\infty]$, *et prenons une suite accouplée* $\{(U_j, V_j)\}$ *définies comme plus haut. Si* $G(x,y)$ *est bornée et intégrable-*L, *et si* $g(-u,-v)$ *est la transformée de Fourier de* $G(x,y)$; *alors on a*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)G(x,y)dxdy = \lim_{U_j,\,V_j\to\infty}\int_{-U_j}^{U_j}\int_{-V_j}^{V_j}\left(1-\frac{|u|}{U_j}\right)\left(1-\frac{|v|}{V_j}\right)\times f(u,v)g(-u,-v)dudv. \tag{7.1}$$

On a, en effet, en vertu de $G(x,y)\in L$,

$$\begin{aligned}&\int_{-U_j}^{U_j}\int_{-V_j}^{V_j}\left(1-\frac{|u|}{U_j}\right)\left(1-\frac{|v|}{V_j}\right)f(u,v)g(-u,-v)dudv\\ &=\frac{1}{2\pi}\int_{-U_j}^{U_j}\int_{-V_j}^{V_j}\left(1-\frac{|u|}{U_j}\right)\left(1-\frac{|v|}{V_j}\right)\\ &\quad\times f(u,v)dudv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y)e^{-i(ux+vy)}dxdy\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y)dxdy\\ &\quad\times\int_{-U_j}^{U}\int_{-V_j}^{V_j}\left(1-\frac{|u|}{U_j}\right)\left(1-\frac{|v|}{V_j}\right)f(u,v)e^{-i(xu+yv)}dudv\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y;\,U_j,V_j)G(x,y)dxdy,\end{aligned}$$

car on voit ici que le changement d'ordre de l'intégration est juste. Or la fonction $G(x,y)$ est, par hypothèse, bornée et intégrable$-L$, donc elle est une fonction de la classe L^q, $\left(\text{où } q=\frac{p}{1-p}\right)$. D'où l'on peut tirer l'égalité (7.1) en vertu du théorème du dernier numéro et de la convergence générale de MM. Hobson-Lebesgue. Il est aussi à remarquer ici qu'on doit utiliser (5.6) dans le cas de $\mathfrak{H}^1$.

8. *Soit* $f(x,y)\in\mathfrak{H}^p[1\leqslant p\leqslant\infty]$. *Alors l'intégrale double*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v)e^{-i(xu+yv)}dudv \tag{8.1}$$

est sommable$-(C,1,1)$ *presque partout à la fonction* $F(x,y)$ *qui est la transformée de Fourier, dans* L^p, *de la fonction* $f(x,y)$.

Prenons, en effet, dans le corollaire du dernier numéro,

$$g(u,v)=\begin{cases}\dfrac{1}{2\pi}\left(1-\dfrac{|u|}{U}\right)\left(1-\dfrac{|v|}{V}\right)e^{i(xu+ry)} & \begin{pmatrix}0\leqslant|u|\leqslant U\\ 0\leqslant|v|\leqslant V\end{pmatrix},\\ 0 & \text{(ailleurs)},\end{cases}$$

alors on a

$$G(\xi,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(u,v)e^{-i(\xi u+\eta v)}dudv$$
$$=\frac{1}{4\pi^2}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)e^{i\{(x-\xi)u+(y-\eta)v\}}dudv$$
$$=\left\{\frac{1}{2\pi}\int_{-U}^{U}\left(1-\frac{|u|}{U}\right)e^{i(x-\xi)u}du\right\}$$
$$\times\left\{\frac{1}{2\pi}\int_{-V}^{V}\left(1-\frac{|v|}{V}\right)e^{i(y-\eta)v}dv\right\}$$
$$=\frac{4}{\pi^2UV}\cdot\frac{\sin^2\frac{1}{2}U(x-\xi)}{(x-\xi)^2}\cdot\frac{\sin^2\frac{1}{2}V(y-\eta)}{(y-\eta)^2}.$$

Cette fonction $G(\xi,\eta)$ satisfait donc à la condition du corollaire du dernier numéro, et par suite elle est bornée et intégrable dans le domaine $\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$. On a par conséquent, grâce au corollaire,

$$\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u,v)e^{-i(xu+yv)}dudv$$
$$=\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}U(x-\xi)}{(x-\xi)^2}\cdot\frac{\sin^2\frac{1}{2}V(y-\eta)}{(y-\eta)^2}$$
$$\times F(\xi,\eta)d\xi d\eta,$$

où la fonction $F(\xi,\eta)$ est définie comme dans le théorème du n° 6. Le second membre de cette égalité est l'intégrale double de M. Fejér, et tend, comme on le sait, presque partout vers la fonction $F(x,y)$ lorsque U et V tendent vers l'infini. On a donc presque partout

$$F(x,y)=\frac{1}{2\pi}\lim_{U,V\to\infty}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)$$
$$\times f(u,v)e^{-i(xu+yv)}dudv,$$

la convergence étant, bien entendu, dans L^p. Et par suite, on a

$$F(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v)e^{-i(xu+yv)}dudv.$$

$(C,1,1)$, presque partout, dans L^p. Ce qui prouve notre proposition.

9. Démontrons maintenant le théorème suivant :

Si $f(x,y)\in\mathfrak{H}^p$ $[2\leqslant p\leqslant\infty]$, *si* $G(x,y)\in L^q\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$, *et si* $g(-x,-y)$ *est la transformée de Fourier, dans* L^p, *de la fonction* $G(x,y)$, *et si, en plus,* $g(x,y)$ *est bornée dans tout domaine fini*; *alors on a*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)\,G(x,y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)g(-x,-y)dxdy, \tag{9.1}$$

dont le premier membre converge, et dont le second est sommable $-(C,1,1)$.

Comme on a, en effet, $f(x, y) \in \mathfrak{H}^p$ et $G(x, y) \in L^p$ par hypothèse, on voit, grâce à la proposition du dernier numéro, que

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y)F(x, y)dxdy = \frac{1}{2\pi}\lim_{U, V\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y)dxdy \times \int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u, v)e^{-i(xu+yv)}dudv;$$

et par suite, en vertu du théorème résultant de la sommation de Cesàro, que

$$= \lim_{U, V\to\infty}\frac{1}{2\pi}\lim_{X, Y\to\infty}\int_{-X}^{X}\int_{-Y}^{Y}\left(1-\frac{|x|}{X}\right)\left(1-\frac{|y|}{Y}\right)G(x, y)dxdy \times \int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u, v)e^{-i(xu+yv)}dudv$$

$$= \lim_{U, V\to\infty} I, \text{ posée, pour abréger.}$$

Or on sait encore par hypothèse que $G(x, y) \in L^q\begin{pmatrix}-\infty, \infty\\ -\infty, \infty\end{pmatrix}$ $[1 \leqslant q \leqslant 2]$. Il existe donc la transformée $g(-u, -v)$ de Fourier, dans L^p, de la fonction $G(x, y)$, pouvant s'écrire de la manière suivante :

$$(9.2)\qquad g(-u, -v) = \frac{1}{2\pi}\lim_{X, Y\to\infty}\int_{-X}^{X}\int_{-Y}^{Y}\left(1-\frac{|x|}{X}\right)\left(1-\frac{|y|}{Y}\right) \times G(x, y)e^{-i(ux+vy)}dxdy,$$

où la convergence est, bien entendu, bornée dans tout domaine où la fonction $g(-u, -v)$ est bornée, c'est-à-dire, dans tout domaine fini $\begin{pmatrix}|u| \leqslant U\\ |v| \leqslant V\end{pmatrix}$. D'où l'on tire

$$I = \frac{1}{2\pi}\lim_{X, Y\to\infty}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u, v)dudv \times \int_{-X}^{X}\int_{-Y}^{Y}\left(1-\frac{|x|}{X}\right)\left(1-\frac{|y|}{Y}\right)G(x, y)e^{-i(ux+vy)}dxdy$$

$$= \int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u, v)g(-u, -v)dudv.$$

On a donc l'égalité suivante :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)G(x, y)dxdy = \lim_{U, V\to\infty}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u, v)g(-u, -v)dudv,$$

qui donne l'égalité (9.1). Notre théorème est ainsi démontré.

10. Nous pouvons démontrer de même le théorème suivant :

Si $f(x, y) \in \mathfrak{H}^p$ $[1 \leqslant p \leqslant 2]$, *si* $g(x, y) \in L^p\begin{pmatrix}-\infty, \infty\\ -\infty, \infty\end{pmatrix}$ *est bornée dans*

tout domaine fini, et si $G(x,y)$ *est la transformée de Fourier, dans* L^q, *de la fonction* $g(x,y)$; *alors on a l'égalité suivante* :

$$(10.1)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)G(x,y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g(-x,-y)\,dxdy,$$

dont le premier membre converge, et dont le second est sommable—$(C,1,1)$.

Comme on a, en effet, $g(x,y)\in L^p\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}[1\leqslant p\leqslant 2]$ par hypotèse, il existe la ttansformée $G(x,y)$ de Fourier, dans L^q, de cette fonction $g(u,v)$. Or on a (9.2), où la convergence est bornée dans tout domaine fini où la fonction $g(-u,-v)$ l'est. On peut donc avoir l'égalité (10.1) de même.

11. Les deux derniers théorèmes donnent le suivant :

Si $f(x,y)\in \mathfrak{H}^p[1\leqslant p\leqslant\infty]$, *et si* $F(x,y)$ *est définie comme dans* $n^\circ 7$; *alors on a la relation suivante* :

$$(11.1)\qquad f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i(xu+yv)}dudv,$$

$(C,1,1)$, *presque partout.*

Prenons, en effet, dans les théorèmes des deux derniers numéros,

$$g(u,v)=\begin{cases}1, & \begin{pmatrix}-x\leqslant u\leqslant 0\\-y\leqslant v\leqslant 0\end{pmatrix},\\ 0, & (\text{ailleurs}),\end{cases}$$

alors on a

$$(11.2)\qquad G(\xi,\eta)=\frac{1}{2\pi}\int_{-x}^{0}\int_{-y}^{0} e^{-i(\xi u+\eta v)}dudv$$

$$=\frac{1}{2\pi}\cdot\frac{e^{i\xi x}-1}{i\xi}\cdot\frac{e^{i\eta y}-1}{i\eta}$$

$$=\frac{2}{\pi}\cdot\frac{\sin\frac{1}{2}\xi x}{\xi}\cdot\frac{\sin\frac{1}{2}\eta y}{\eta}e^{i\frac{\xi x+\eta y}{2}}.$$

Toutes les hypothèses des théorèmes des deux derniers numéros sont donc satisfaites. Il en résulte que

$$(11.3)\qquad \int_0^x\int_0^y f(u,v)dudv=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\frac{1}{2}\xi x}{\xi}\cdot\frac{\sin\frac{1}{2}\eta y}{\eta}$$

$$\times F(\xi,\eta)e^{i\frac{x\xi+y\eta}{2}}d\xi d\eta.$$

Premier cas : $p=1$. On sait que $F(\xi,\eta)\in L\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$, et de plus, que le second membre de l'égalité (11.3) peut s'écrire, grâce à (11.2),

$$\frac{1}{2\pi}\int_0^x\int_0^y dudv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi,\eta)e^{i(u\xi+v\eta)}d\xi d\eta\,;$$

ce qui donne immédiatement l'égalité (11.1).

Deuxième cas : $p>1$. On a, dans ce cas, en vertu de l'inégalité généralisée de Hölder et de l'égalité (11.3),

$$\begin{aligned}
&\left|\int_0^x\int_0^y f(u,v)\,du\,dv\right| \\
&\leqslant \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}\xi x}{\xi}\cdot\frac{\sin\frac{1}{2}\eta y}{\eta}\right|^q d\xi\,d\eta\right]^{\frac{1}{q}} \\
&\quad\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(\xi,\eta)|^p d\xi\,d\eta\right]^{\frac{1}{p}} \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{x\sin\theta}{2\theta}\cdot\frac{y\sin\phi}{2\phi}\right|^q \frac{2\,d\theta}{x}\cdot\frac{2\,d\phi}{y}\right]^{\frac{1}{q}} \qquad (11.4)\\
&\quad\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(\xi,\eta)|^p d\xi\,d\eta\right]^{\frac{1}{p}} \\
&\leqslant \frac{2M}{\pi}\left\{\frac{(xy)^{q-1}}{2^{2(q-1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\theta}{\theta}\cdot\frac{\sin\phi}{\phi}\right|^q d\theta\,d\phi\right\}^{\frac{1}{q}} \\
&\leqslant \frac{2}{\pi}M(xy)^{\frac{1}{p}}\left\{\frac{1}{2^{2(q-1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\theta}{\theta}\cdot\frac{\sin\phi}{\phi}\right|^q d\theta\,d\phi\right\}^{\frac{1}{q}} \leqslant M(xy)^{\frac{1}{p}}.
\end{aligned}$$

Si l'on prend

$$G(u,v)=\begin{cases}\dfrac{1}{2\pi}\left(1-\dfrac{|u|}{U}\right)\left(1-\dfrac{|v|}{V}\right)e^{i(xu+yv)}, & \left(\begin{matrix}|u|\leqslant U\\ |v|\leqslant V\end{matrix}\right),\\ 0. & \text{(ailleurs)},\end{cases}$$

on a

$$\begin{aligned}
g(-\xi,-\eta)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(u,v)e^{-i(\xi u+\eta v)}\,du\,dv\\
&=\frac{1}{4\pi^2}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)\\
&\quad\times e^{i\{(x-\xi)u+(y-\eta)v\}}\,du\,dv\\
&=\frac{1}{\pi^2UV}\cdot\frac{1-\cos U(x-\xi)}{(x-\xi)^2}\cdot\frac{1-\cos V(y-\eta)}{(y-\eta)^2}\\
&=\frac{4}{\pi^2UV}\cdot\frac{\sin^2\frac{1}{2}U(x-\xi)}{(x-\xi)^2}\cdot\frac{\sin^2\frac{1}{2}V(y-\eta)}{(y-\eta)^2},
\end{aligned}$$

et par suite toutes les hypothèses des théorèmes des deux derniers numéros sont aussi satisfaites. Il en résulte que

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{U}\right)F(u,v)e^{i(xu+yv)}\,du\,dv\\
&=\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\frac{\sin^2\frac{1}{2}U(x-\xi)}{(x-\xi)^2}\cdot\frac{\sin^2\frac{1}{2}V(y-\eta)}{(y-\eta)^2}\,d\xi\,d\eta,\\
&\qquad\qquad (C,1,1),\\
&=\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x+\xi,y+\eta)\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\cdot\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}\,d\xi\,d\eta, \quad (C,1,1).
\end{aligned}$$

Or, l'intégrale double du dernier membre existe comme l'intégrale double impropre de M. Lebesgue. En posant, en effet,

$$f_{1,1}(\xi,\eta)=\int_0^{\xi}\int_0^{\eta} f(u,v)\,du\,dv,$$

on a

$$\begin{aligned}
&\int_0^X\int_0^Y f(x+\xi,y+\eta)\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi\,d\eta\\
&=f_{1,1}(x+X,y+Y)\frac{\sin^2\frac{1}{2}UX}{X^2}\frac{\sin^2\frac{1}{2}VY}{Y^2}\\
&\quad-f_{1,1}(x,y+Y)\frac{U^2}{4}\frac{\sin^2\frac{1}{2}VY}{Y^2}\\
&\quad-f_{1,1}(x+X,y)\frac{\sin^2\frac{1}{2}UX}{X^2}\frac{V^2}{4}+f_{1,1}(x,y)\frac{U^2}{4}\frac{V^2}{4}\\
&\quad-\frac{\sin^2\frac{1}{2}VY}{Y^2}\int_0^X f_{1,1}(x+\xi,y+Y)\frac{d}{d\xi}\left(\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\right)d\xi\\
&\quad-\frac{\sin^2\frac{1}{2}UX}{X^2}\int_0^Y f_{1,1}(x+X,y+\eta)\frac{d}{d\eta}\left(\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}\right)d\eta\\
&\quad+\frac{V^2}{4}\int_0^X f_{1,1}(x+\xi,y)\frac{d}{d\xi}\left(\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\right)d\xi\\
&\quad+\frac{U^2}{4}\int_0^Y f_{1,1}(x,y+\eta)\frac{d}{d\eta}\left(\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}\right)d\eta\\
&\quad-\int_0^X\int_0^Y f_{1,1}(x+\xi,y+\eta)\frac{d}{d\xi}\left(\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\right)\frac{d}{d\eta}\left(\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}\right)d\xi\,d\eta,
\end{aligned}$$

et l'on voit facilement que cette dernière expression tend vers une limite lorsque X et Y tendent vers l'infini, en vertu de (11.4).

Désignons maintenant par k_1 et k_2 deux constantes positives satisfaisant aux relations $\begin{matrix}|x|\leqslant k_1\\ |y|\leqslant k_2\end{matrix}$. L'intégrale double du dernier membre de (11.5) peut alors s'écrire sous la forme suivante :

$$\begin{aligned}
&\frac{4}{\pi^2UV}\left\{\int_{-k_1}^{k_1}\int_{-k_2}^{k_2}+\int_{k_1}^{\infty}\int_{k_2}^{\infty}+\int_{-\infty}^{-k_1}\int_{k_2}^{\infty}+\int_{-\infty}^{-k_1}\int_{-\infty}^{-k_2}+\int_{k_1}^{\infty}\int_{-\infty}^{-k_2}\right.\\
&\left.\quad+\int_{k_1}^{\infty}\int_{-k_2}^{k_2}+\int_{-k_1}^{k_1}\int_{k_2}^{\infty}+\int_{-\infty}^{-k_1}\int_{-k_2}^{k_2}+\int_{-k_1}^{k_1}\int_{-\infty}^{-k_2}\right\}=\sum_{j=1}^{9}I_j.
\end{aligned}$$

On a, d'abord,

$$\begin{aligned}
I_2&=\frac{4}{\pi^2UV}\int_{k_1}^{\infty}\int_{k_2}^{\infty}f(x+\xi,y+\eta)\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi\,d\eta\\
&=\frac{4\sin^2\frac{1}{2}Uk_1\sin^2\frac{1}{2}Vk_2}{\pi^2UVk_1^2k_2^2}f_{1,1}(x+k_1,y+k_2)\\
&\quad+\frac{4\sin^2\frac{1}{2}Vk_2}{\pi^2UVk_2^2}\int_{k_1}^{\infty}f_{1,1}(x+\xi,y+k_2)\\
&\quad\times\left(\frac{-2\sin^2\frac{1}{2}U\xi}{\xi^3}+\frac{U\sin U\xi}{2\xi^2}\right)d\xi
\end{aligned}$$

$$+\frac{4\sin^2\frac{1}{2}U k_1}{\pi^2 U V k_1^2}\int_{k_2}^{\infty} f_{1,1}(x+k_1, y+\eta)\times\left(\frac{-2\sin^2\frac{1}{2}V\eta}{\eta^3}+\frac{V\sin V\eta}{2\eta^2}\right)d\eta$$

$$+\frac{4}{\pi^2 UV}\int_{k_1}^{\infty}\int_{k_2}^{\infty} f_{1,1}(x+\xi, y+\eta)\times\left(\frac{-2\sin^2\frac{1}{2}U\xi}{\xi^3}+\frac{U\sin U\xi}{2\xi^2}\right)\times\left(\frac{-2\sin^2\frac{1}{2}V\eta}{\eta^3}+\frac{V\sin V\eta}{2\eta^2}\right)d\xi d\eta.$$

Or on sait, grâce à (11.4), que $|f_{1,1}(x+\xi, y+\eta)|\leqslant M|(x+\xi)(y+\eta)|^{\frac{1}{p}}$. Donc on a, en vertu du théorème de M. Lebesgue et Riemann, pour les valeurs finies fixes de k_1 et k_2,

$$\lim_{U, V\to\infty} I_2=0.$$

On a de même, $\lim_{U, V\to\infty} I_3=\lim_{U, V\to\infty} I_4=\lim_{U, V\to\infty} I_5=0$.

On a, ensuite,

$$I_6=\frac{4}{\pi^2 UV}\int_{k_1}^{\infty}\int_{-k_2}^{k_2} f(x+\xi, y+\eta)\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi d\eta$$

$$=\frac{4\sin^2\frac{1}{2}Uk_1\sin^2\frac{1}{2}Vk_2}{\pi^2 UVk_1^2k_2^2}[f_{1,1}(x+k_1, y-k_2)-f_{1,1}(x+k_1, y+k_2)]$$

$$+\frac{4\sin^2\frac{1}{2}Vk_2}{\pi^2 UVk_2^2}\int_{k_1}^{\infty}[f_{1,1}(x+\xi, y-k_2)-f_{1,1}(x+\xi, y+k_2)]\times\left(\frac{-2\sin^2\frac{1}{2}U\xi}{\xi^3}+\frac{U\sin U\xi}{2\xi^2}\right)d\xi$$

$$+\frac{4\sin^2\frac{1}{2}Uk_1}{\pi^2 UVk_1^2}\int_{-k_2}^{k_2} f_{1,1}(x+k_1, y+\eta)\times\left(\frac{-2\sin^2\frac{1}{2}V\eta}{\eta^3}+\frac{V\sin V\eta}{2\eta^2}\right)d\eta$$

$$+\frac{4}{\pi^2 UV}\int_{k_1}^{\infty}\int_{-k_2}^{k_2} f_{1,1}(x+\xi, y+\eta)\times\left(\frac{-2\sin^2\frac{1}{2}U\xi}{\xi^3}+\frac{U\sin U\xi}{2\xi^2}\right)\times\left(\frac{-2\sin^2\frac{1}{2}V\eta}{\eta^3}+\frac{V\sin V\eta}{2\eta^2}\right)d\xi d\eta,$$

d'où l'on tire aussi

$$\lim_{U, V\to\infty} I_6=0.$$

On a, de même,

$$\lim_{U, V\to\infty} I_7=\lim_{U, V\to\infty} I_8=\lim_{U, V\to\infty} I_9=0.$$

On sait finalement que I_1 est l'intégrale double de M. Fejér, et l'on a

$$\lim_{U,V\to\infty} I_1 = f(x,y), \text{ presque partout.}$$

Par conséquent, on a

$$\lim_{U,V\to\infty} \frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right) \times F(u,v)e^{i(xu+yv)}dudv = f(x\ y), \qquad \text{presque partout.}$$

C'est-à-dire, on a

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{i(xu+yv)}dudv,$$

$(C,1,1)$, presque partout.

Notre théorème est ainsi démontré.

12. Nous avons donc, en vertu du théorème du $n^\circ 8$ et de celui du dernier numéro, le théorème fondamental suivant dans la classe $\mathfrak{H}^p$ $[1\leqslant p\leqslant\infty]$:

Si $f(x,y)\in\mathfrak{H}^p$ $[1\leqslant p\leqslant\infty]$, *alors*

$$(12.1)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v)e^{-i(xu+yv)}dudv\to F(x,y)\in L^p\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix},$$

$(C,1,1)$, *presque partout, et cette fonction* $F(x,y)$ *est la transformée de Fourier, dans* L^p, *de* $f(x,y)$, *et l'on a*

$$(12.2)\qquad f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{i(xu+yv)}dudv,$$

$(C,1,1)$. *presque partout. Il est à remarquer ici que la fonction* $F(x,y)$ *est bornée dans le cas* $\mathfrak{H}^\infty$.

§ 2. La classe $\mathfrak{H}^{*p}$

13. Soit $F(x,y)$ une fonction mesurable et appartenant à la classe $L^p\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$ $[1\leqslant p<\infty]$, et si, en plus, l'intégrale double

$$(13.1)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{iux}-1}{iu}\frac{e^{ivy}-1}{iv}F(u,v)dudv=I_1$$

existe comme une intégrale-L indéfinie ; alors on dit que la fonction $F(x,y)$ appartient à *la classe* $\mathfrak{H}^{*p}[1\leqslant p<\infty]$. Soit $F(x,y)$ une fonction bornée, et si la somme

$$(13.2)\qquad \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{e^{ixu}-1}{iu}\frac{e^{iyv}-1}{iv}F(u,v)dudv + \frac{1}{2\pi}\left(\int_{1}^{\infty}\int_{-1}^{1}+\int_{-\infty}^{-1}\int_{-1}^{1}\right)\frac{e^{ixu}}{iu}\frac{e^{iyv}-1}{iv}F(u,v)dudv$$

$$+\frac{1}{2\pi}\left(\int_{-1}^{1}\int_{1}^{\infty}+\int_{-1}^{1}\int_{-\infty}^{-1}\right)\frac{e^{ixu}-1}{iu}\frac{e^{iyv}}{iv}F(u,v)dudv$$
$$+\frac{1}{2\pi}\left(\int_{1}^{\infty}\int_{1}^{\infty}+\int_{-\infty}^{-1}\int_{1}^{\infty}+\int_{1}^{\infty}\int_{-\infty}^{-1}+\int_{-\infty}^{-1}\int_{-\infty}^{-1}\right)$$
$$\frac{e^{ixu}}{iu}\frac{e^{iyv}}{iv}F(u,v)dudv=I_2$$

est sommable-$(C,1,1)$ et si cette somme existe comme une intégrale-L indéfinie ; alors on dit que la fonction $F(x,y)$ appartient à *la classe* $\mathfrak{H}^{*x}$.

Démontrons maintenant le théorème suivant :

Si $F(x,y)\in\mathfrak{H}^{*p}[1\leqslant p\leqslant\infty]$, *et si*

$$(13.3)\qquad f(x,y)=\frac{\partial^2 I_1}{\partial x\partial y}\qquad(\textit{pour } 1\leqslant p<\infty),$$
$$=\frac{\partial^2 I_2}{\partial x\partial y}\qquad(\textit{pour } p=\infty),$$

alors la fonction $f(x,y)$ *appartient à la classe* $\mathfrak{H}^p[1\leqslant p\leqslant\infty]$, *et la fonction* $F(x,y)$ *est la transformée de Fourier, dans* L^p, *de la fonction* $f(x,y)$.

Considérons tout d'abord le cas où $1\leqslant p<\infty$. Comme on a $F(x,y)\in\mathfrak{H}^{*p}$ par hypothèse, l'intégrale double I_1 existe comme une intégrale-L indéfinie, et donc elle converge uniformément dans tout domaine fini. On a donc

$$F(x,y;U,V)=\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)$$
$$\times f(u,v)e^{-i(xu+yv)}dudv$$
$$=\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\frac{d}{du}\left\{\left(1-\frac{|u|}{U}\right)e^{-ixu}\right\}\frac{d}{dv}\left\{\left(1-\frac{|v|}{V}\right)e^{-iyv}\right\}$$
$$\times f_{1,1}(u,v)dudv$$
$$=\frac{1}{4\pi^2}\int_{-U}^{U}\int_{-V}^{V}\frac{d}{du}\left\{\left(1-\frac{|u|}{U}\right)e^{-ixu}\right\}\frac{d}{dv}\left\{\left(1-\frac{|v|}{V}\right)e^{-iyv}\right\}dudv$$
$$(13.4)\qquad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ix\xi}-1}{i\xi}\frac{e^{iy\eta}-1}{i\eta}F(\xi,\eta)d\xi d\eta$$
$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi,\eta)d\xi d\eta\int_{-U}^{U}\int_{-V}^{V}\frac{d}{du}\left\{\left(1-\frac{|u|}{U}\right)e^{-ixu}\right\}$$
$$\times\frac{d}{dv}\left\{\left(1-\frac{|v|}{V}\right)e^{-iyv}\right\}\frac{e^{ix\xi}-1}{i\xi}\frac{e^{iy\eta}-1}{i\eta}dudv$$
$$=\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi,\eta)\frac{\sin^2\frac{1}{2}U(\xi-x)}{(\xi-x)^2}\cdot\frac{\sin^2\frac{1}{2}V(\eta-y)}{(\eta-y)^2}d\xi d\eta$$
$$=\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi+x,\eta+y)\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\cdot\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi d\eta.$$

Pour le cas où $p=1$, on a

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y;\ U,V)|dxdy$$
$$\leqslant\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi d\eta$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(\xi+x,\eta+y)|dxdy\leqslant M;$$

et l'on a, $\mathfrak{A}$ désignant un ensemble quelconque,

$$\iint_{\mathfrak{A}}|F(x,y;\ U,V)|dxdy$$
$$\leqslant\frac{4}{\pi^2UV}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}U\xi}{\xi^2}\frac{\sin^2\frac{1}{2}V\eta}{\eta^2}d\xi d\eta$$
$$\times\iint_{\mathfrak{A}}|F(\xi+x,\eta+y)|dxdy.$$

Or on sait, par hypothèse, pue $F(x,y)\in L\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$. On peut donc, pour $\varepsilon(>0)$ assez petite donnée à l'avance, trouver une quantité δ, indépendante de ξ et de η, telle qu'on ait

$$\iint_{\mathfrak{A}}|F(\xi+x,\eta+y)|dxdy\leqslant\varepsilon,\qquad\text{si } m(\mathfrak{A})\leqslant\delta.$$

Il en résulte que

$$\iint_{\mathfrak{A}}|F(x,y;\ U,V)|dxdy\leqslant\varepsilon,$$

ce qui prouve que $f(x,y)\in\mathfrak{H}^1$.

Pour le cas où $1<p<\infty$, on a, grâce à l'inégalité généralisée de Hölder,

$$|F(x,y;\ U,V)|\leqslant\frac{1}{\pi^2UV}$$
$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}U\xi}{\xi}\frac{\sin\frac{1}{2}V\eta}{\eta}F(\xi+x,\eta+y)\right|^p d\xi d\eta\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}U\xi}{\xi}\frac{\sin\frac{1}{2}V\eta}{\eta}\right|^q d\xi d\eta\right\}^{\frac{1}{q}}\qquad\left(q=\frac{p}{p-1}\right)$$
$$\leqslant\frac{N}{(UV)^{\frac{1}{q}}}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}U\xi}{\xi}\frac{\sin\frac{1}{2}V\eta}{\eta}\right.\right.$$
$$\left.\left.\times F(\xi+x,\eta+y)\right|^p d\xi d\eta\right\}^{\frac{1}{p}};$$

ce qui donne

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y;\ U,V)|^p dxdy$$
$$\leqslant\frac{M}{|UV|^{p-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x+\xi,y+\eta)|^p dxdy$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\sin \frac{1}{2}U\xi}{\xi} \frac{\sin \frac{1}{2}V\eta}{\eta} \right|^p d\xi d\eta ;$$

d'où l'on tire

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x, y ; U, V)|^p dx dy \leqslant M,$$

ce que prouve que $f(x, y) \in \mathfrak{H}^p [1 < p < \infty]$.

On voit ainsi que $f(x, y)$ appartient à la classe $\mathfrak{H}^p [1 \leqslant p < \infty]$. Il reste donc à affirmer que la fonction $F(x, y)$ est la transformée de Fourier, dans L^p, de la fonction $f(x, y)$. Il suffit, pour cela, de voir que

$$F(x, y ; U, V) \to F(x, y), \qquad \text{presque partout ;}$$

et ceci est manifeste en vertu de (13.4).

Si l'on utilise I_2 au lieu de I_1, on peut alors affirmer que l'intégrale correspondante converge aussi par un argument analogue, et que $f(x, y) \in \mathfrak{H}^\infty$.

Notre théorème est donc démontré.

14. *Si* $f(x, y) \in \mathfrak{H}^p$ $[1 \leqslant p \leqslant \infty]$, *alors* $F(x, y) \in \mathfrak{H}^{*p}$ $[1 \leqslant p \leqslant \infty]$.

On a, en effet, dans le cas où $1 \leqslant p < \infty$,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu} \frac{e^{iyv}-1}{iv} F(u, v) du dv \\ &= \frac{1}{4\pi^2} \lim_{U, V \to \infty} \int_{-U}^{U}\int_{-V}^{V} \left(1 - \frac{|u|}{U}\right)\left(1 - \frac{|v|}{V}\right) f(\xi, \eta) d\xi d\eta \\ &\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu} \frac{e^{iyv}-1}{iv} e^{-i(\xi u + \eta v)} du dv. \end{aligned} \tag{14.1}$$

Or

$$\begin{aligned} f(\xi, \eta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) e^{i(\xi u + \eta v)} du dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) \frac{d}{d\xi}\left(\frac{e^{i\xi u}}{iu}\right) \frac{d}{d\eta}\left(\frac{e^{i\eta v}}{iv}\right) du dv, \end{aligned}$$

et l'on a

$$\begin{aligned} &\int_0^x \int_0^y f(\xi, \eta) d\xi d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) \frac{e^{ixu}-1}{iu} \frac{e^{iyv}-1}{iv} du dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu} \frac{e^{iyv}-1}{iv} \\ &\quad \times \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s, t) e^{-i(us+vt)} ds dt \right\} du dv \\ &= \frac{1}{4\pi^2} \lim_{U, V \to \infty} \int_{-U}^{U}\int_{-V}^{V} \left(1 - \frac{|s|}{U}\right)\left(1 - \frac{|t|}{V}\right) f(s, t) ds dt \end{aligned}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ixu}-1}{iu}\frac{e^{iyv}-1}{iv}e^{-i(xu+yv)}dudv.$$

On voit donc que l'expression (14.1) est égale à $\int_0^x\int_0^y f(\xi,\eta)d\xi d\eta$. Il en résulte que $F(x,y)\in\mathfrak{H}^{*p}$ $[1\leqslant p<\infty]$.

On peut aussi affirmer que cette proposition est réalisée pour le cas où $p=\infty$.

15. Il est aussi à remarquer la proposition suivante :

Soit $F(x,y)\in L^p\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$ $[1\leqslant p\leqslant\infty]$, *et si l'intégrale double*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{i(xu+yv)}dudv \tag{15.1}$$

converge partout vers $f(x,y)$ *qui est finie partout et qui est intégrable-L dans tout domaine fini ; alors on a* $F(x,y)\in\mathfrak{H}^{*p}$, *et cette fonction* $F(x,y)$ *est la transformée de Fourier, dans* L^p, *de la fonction* $f(x,y)$.

On a, en effet, grâce à $F(x,y)\in L^p$, par une part de l'hypothèse,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|F(x,y)|}{(1+x^2)(1+y^2)}dxdy\\&\leqslant\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x,y)|^p dxdy\right\}^{\frac{1}{p}}\\&\quad\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{1}{(1+x^2)(1+y^2)}\right|^q dxdy\right\}^{\frac{1}{q}}\leqslant M.\end{aligned} \tag{15.2}$$

En utilisant donc l'autre part de l'hypothèse, on peut, avec M. Pollard, affirmer que

$$\begin{aligned}&\int_0^x\int_0^y dsdt\int_0^s\int_0^t f(u,v)dudv\\&=\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{e^{ixu}-ixu-1}{u^2}\frac{e^{iyv}-iyv-1}{v^2}F(u,v)dudv\\&+\frac{1}{2\pi}\left(\int_1^{\infty}\int_{-1}^{1}+\int_{-\infty}^{-1}\int_{-1}^{1}\right)\frac{e^{ixu}}{u^2}\frac{e^{iyv}-iyv-1}{v^2}F(u,v)dudv\\&+\frac{1}{2\pi}\left(\int_{-1}^{1}\int_1^{\infty}+\int_{-1}^{1}\int_{-\infty}^{-1}\right)\frac{e^{ixu}-ixu-1}{u^2}\frac{e^{iyv}}{v^2}F(u,v)dudv\\&+\frac{1}{2\pi}\left(\int_1^{\infty}\int_1^{\infty}+\int_{-\infty}^{-1}\int_1^{\infty}+\int_1^{\infty}\int_{-\infty}^{-1}+\int_{-\infty}^{-1}\int_{-\infty}^{-1}\right)\frac{e^{ixu}}{u^2}\frac{e^{iyv}}{v^2}F(u,v)dudv\\&+\text{termes finis.}\end{aligned}$$

Il en résulte que

$$f(x,y)=\frac{1}{2\pi}\frac{\partial^2}{\partial x\partial y}\int_{-8}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ixu}-1}{iu}\frac{e^{iyv}-1}{iv}F(u,v)dudv,$$

ce qui prouve notre proposition.

§ 3. La classe L^{*p}

16. Désignons, pour abréger, par $f(x,y)\in\mathfrak{H}^p\,L^{p'}$ $[1\leqslant p\leqslant\infty,\ 1\leqslant p'\leqslant\infty]$ le fait associé $f(x,y)\in\mathfrak{H}^p$ et $f(x,y)\in L^{p'}$, et par L^{*p} la classe $\mathfrak{H}^pL^p$.

Pour la classe $\mathfrak{H}^pL^{p'}$ démontrons le théorème suivant :

Si $f(x,y)\in\mathfrak{H}^pL^{p'}$ $[1\leqslant p\leqslant\infty,\ 1\leqslant p'\leqslant\infty]$, *alors la fonction* $f(x,y)$ *a sa transformée de Fourier* $F(x,y)$ *dans* L^p, *et cette fonction* $F(x,y)$ *appartient à la classe* $\mathfrak{H}^{p'}L^p$, *et de plus, la fonction* $f(-x,-y)$ *est la transformée de Fourier, dans* $L^{p'}$, *de la fonction* $F(x,y)$.

Considérons, en effet, tout d'abord le cas où $1\leqslant p<\infty$, $1\leqslant p'\leqslant\infty$. Dans les théorèmes des $n^\circ 9$ et 10, posons

$$G(\xi,\eta)=\begin{cases}1, & \begin{pmatrix}0\leqslant\xi\leqslant x\\ 0\leqslant\eta\leqslant y\end{pmatrix},\\ 0, & \text{(ailleurs)}.\end{cases}$$

Alors on a

$$\begin{aligned}g(-u,-v)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\xi,\eta)e^{-i(u\xi+v\eta)}\,d\xi d\eta\\&=\frac{1}{2\pi}\int_0^x\int_0^y e^{-i(u\xi+v\eta)}d\xi d\eta=\frac{1}{2\pi}\,\frac{1-e^{-ixu}}{iu}\,\frac{1-e^{-iyv}}{iv},\end{aligned}$$

d'où il résulte que

$$\begin{aligned}(16.1)\qquad&\int_0^x\int_0^y F(\xi,\eta)\,d\xi d\eta\\&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1-e^{-ixu}}{iu}\,\frac{1-e^{-iyv}}{iv}f(u,v)dudv.\end{aligned}$$

Or on a, d'une part, $f(x,y)\in L^{p'}$ par hypothèse. On voit donc, grâce à la proposition du dernier numéro, que $f(x,y)\in\mathfrak{H}^{*p'}$. D'où il résulte, en vertu du théorème du $n^\circ 13$, que $F(x,y)\in\mathfrak{H}^{p'}$.

On a, d'autre part, $f(x,y)\in\mathfrak{H}^p$ par hypothèse. On a donc $F(x,y)\in L^p$. On voit ainsi que $F(x,y)\in\mathfrak{H}^{p'}L^p$. Il en résulte que $f(-x,-y)\in L^{p'}$.

Prenons ensuite le cas où $p=\infty$, $1\leqslant p'\leqslant\infty$. Comme on a dans ce cas

$$\begin{aligned}F(x,y)=\frac{1}{2\pi}\lim_{U,V\to\infty}&\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)\\&\times f(u,v)e^{-i(xu+yv)}dudv,\end{aligned}$$

la convergence étant bien entendu bornée, on voit que

$$\int_0^x\int_0^y F(\xi,\eta)d\xi d\eta=\frac{1}{2\pi}\lim_{U,V\to\infty}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|u|}{U}\right)\left(1-\frac{|v|}{V}\right)f(u,v)$$

$$\times \frac{1-e^{-ixu}}{iu}\,\frac{1-e^{-iyv}}{iv}\,dudv.$$

On peut donc retrouver de même le résultat voulu.

Notre théorème est ainsi démontré.

17. On peut en déduire le théorème suivant :

Si $f(x,y)\in L^{*p}$ $[1\leqslant p\leqslant\infty]$, *alors* $F(x,y)\in L^{*p}$.

Ce théorème est fondamental pour la classe L^{*p} parce qu'il y a une propriété réciproque entre la fonction et sa transformée de Fourier. Il y a bien entendu une classe des fonctions réciproques en elles-mêmes, mais nous n'y touchons pas ici. Toutefois, il est à remarquer que $L^{*p}\subset L^p$ pour $1\leqslant p\leqslant 2$, mais qu'il ne s'en suit pas nécessairement que $L^{*p}\subset L^p$ pour $p>2$. Malgré cela on peut affirmer la proposition suivante :

Si $f(x,y)\in\mathfrak{H}^p$ $[1<p<\infty]$, si $F(x,y)$ est la transformée de Fourier, dans L^p, de la fonction $f(x,y)$, et si

$$(17.1)\qquad \mathfrak{F}(x,y;\,U,V)=\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}f(u,v)e^{-i(xu+yv)}dudv\,;$$

alors on a

$$(17.2)\qquad \lim_{U,V\to\infty}\int_{-U}^{U}\int_{-V}^{V}|F(x,y)-\mathfrak{F}(x,y;\,U,V)|^p dxdy=0,$$

ce qui affirme que la fonction $F(x,y)$ est aussi la transformée ordinaire de Fourier, dans L^p, de $f(x,y)$. Inversement, si la fonction $f(x,y)$ satisfait à la relation (17.2), alors $f(x,y)\in\mathfrak{H}^p$, et la fonction $F(x,y)$ est la transformée, au sens jusqu'à présent, de Fourier, dans L^p de la fonction $f(x,y)$.

III. La Quasi-analyticité faisant usage des transformées de Fourier

§ 1. Une classe quasi-analytique dans L^{*p}

18. Soit $\phi(u,v)$ une fonction positive, dérivable dans le domaine $\binom{0,\infty}{0,\infty}$ et saisfaisant, pour chaque valeur positive d'un paramètre ν, au condition

$$u\frac{d}{du}\phi(u,\nu u)\uparrow\infty \qquad \text{avec } u\to\infty.$$

Désignons, comme dans le $n^\circ 3$, par $N(u,\nu u)$ la partie entière du produit $u\frac{d\phi(u,\nu u)}{du}$, alors on a

$$\phi(1,\nu)+\int_1^R\frac{N(u,\nu u)}{u}du\leqslant\phi(R,r)$$
$$<\phi(1,\nu)+\log R+\int_1^R\frac{N(u,\nu u)}{u}du,$$

où $r=\nu R$. Or on sait qu'on peut construire une fonction entière $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{m,n}R^m r^n$ (où $C_{m,n}\geqq 0$ et où $\lim_{m+n\to\infty} \sqrt[m+n]{C_{m,n}}=0$), telle qu'on ait

$$\log T(R,r)=\int_1^R \frac{N(u,\nu u)}{u}du,$$

où $T(R,r)=\max_{m,n\geq 0} C_{m,n}R^m r^n = C_{\lambda,\mu}R^\lambda r^\mu$, $N(R,r)=\lambda(R,r)+\mu(R,r)$.
On a donc

$$\phi(1,\nu)+\log T(R,r)\leqq\phi(R,r)<\phi(1,\nu)+\log\{RT(R,r)\}.$$

On voit donc que deux intégrales doubles

$$\int_1^\infty\int_1^\infty \log T(R,r)\frac{dR}{R^2}\frac{dr}{r^2} \quad\text{et}\quad \int_1^\infty\int_1^\infty \phi(u,v)\frac{du}{u^2}\frac{dv}{v^2}$$

divergent ou convergent en même temps.

Considérons maintenant la fonction $f(x,y)$ appartenant à la classe $L^{*p}\begin{pmatrix}-\infty,\infty\\-\infty,\infty\end{pmatrix}$ $[1\leqq p\leqq\infty]$, indéfiniment dérivable dans ce domaine, et satisfaisant aux conditions suivantes :

(18.1) $\quad f^{(m+n)}(0,0)=0 \qquad \begin{pmatrix}m=0,1,2,\cdots\\ n=0,1,2,\cdots\end{pmatrix}$,

(18.2) $\quad |F(\pm u,\pm v)|\leqq Ae^{-\phi(u,v)} \qquad$ (A : constante),

où $F(u,v)$ est la transformée de Fourier de $f(x,y)$, c'est-à-dire

$$F(u,v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}dxdy.$$

Si l'intégrale double

(18.3) $\quad \int_1^\infty\int_1^\infty \phi(u,v)\frac{du}{u^2}\frac{dv}{v^2}$

diverge, alors la fonction $f(x,y)$ est identiquement nulle.

On a, en effet, d'une part

$$|F(\pm u,\pm v)|\leqq Ae^{-\phi(u,v)}\leqq A\exp\{-\phi(1,\nu)-\log T(u,v)\}$$
$$=\frac{A}{T(u,v)}\leqq\frac{A}{C_{p,q}u^p v^q}$$

pour toutes valeurs positives entières de p et q, et l'on a, d'autre part

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i(xu+yv)}dudv.$$

Donc on a

$$|f^{(m+n)}(x,y)|=\left|\frac{i^{m+n}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)u^m v^n e^{i(xu+yv)}dudv\right|$$
$$\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(u,v)u^m v^n|\,dudv$$
$$\begin{pmatrix}m=0,1,2,\cdots\\ n=0,1,2,\cdots\end{pmatrix}.$$

En prenant ici $p=m+2$, $q=n+2$, on voit que

$$|f^{(m+n)}(x,y)|\leqq\frac{A}{C_{m+2,\,n+2}}\qquad\begin{pmatrix}m=0,\,1,\,2,\,\cdots\\ n=0,\,1,\,2,\,\cdots\end{pmatrix}.$$

Considérons donc la fonction $\psi(x,y)=\int_0^x\int_0^y d\xi d\eta\int_0^\xi\int_0^\eta f(u,v)dudv$. Alors, en posant $M_{m,n}=1/C_{m+2,n+2}\begin{pmatrix}m\geqq 0\\ n\geqq 0\end{pmatrix}$, on voit que cette fonction $\psi(x,y)$ satisfait aux conditions suivantes :

$$|\psi^{(m+n)}(x,y)|\leqq AM_{m,n}\qquad\begin{pmatrix}-\infty\leqq x\leqq\infty\,;\ m=0,\,1,\,2,\,\cdots\\ -\infty\leqq y\leqq\infty\,;\ n=0,\,1,\,2,\,\cdots\end{pmatrix},$$

$$\psi^{(m+n)}(0,0)=0\qquad\begin{pmatrix}m=0,\,1,\,2,\,\cdots\\ n=0,\,1,\,2,\,\cdots\end{pmatrix},$$

et de plus, l'intégrale double $\int_1^\infty\int_1^\infty\log T(R,r)\frac{dR}{R^2}\frac{dr}{r^2}$ diverge. On voit donc, en vertu du théorème extensif de MM. Denjoy et Carleman que la fonction $\psi(x,y)$ est identiquement nulle, ce qui prouve que $f(x,y)$ l'est aussi.

19. Supposons, au contraire, que l'intégrale double (18.3) converge. Alors, comme on le voit plus haut, l'intégrale double

$$\int_1^\infty\int_1^\infty\log T(R,r)\frac{dR}{R^2}\frac{dr}{r^2}$$

converge aussi. On peut donc, en vertu du théorème fondamental du n° 18 de mon Mémoire (I), construire une fonction $f_1(x,y)$, non identiquement nulle, indéfiniment dérivable dans un certain domaine $\begin{pmatrix}a\leqq x\leqq a_1\\ b\leqq y\leqq b_1\end{pmatrix}$, non négative et vérifiant les relations

$$f_1^{(m+n)}(a,b)=f_1^{(m+n)}(a,b_1)=f_1^{(m+n)}(a_1,b)$$
$$=f_1^{(m+n)}(a_1,b_1)=0\qquad\begin{pmatrix}m=0,\,1,\,2,\,\cdots\\ n=0,\,1,\,2,\,\cdots\end{pmatrix},$$

$$|f_1^{(m+n)}(x,y)|<M_{m,n}\qquad\begin{pmatrix}a\leqq x\leqq a_1\,;\ m=0,\,1,\,2,\,\cdots\\ b\leqq y\leqq b_1\,;\ n=0,\,1,\,2,\,\cdots\end{pmatrix}.$$

Posons donc

$$f(x,y)=\begin{cases}f_1(x,y), & \begin{pmatrix}a\leqq x\leqq a_1\\ b\leqq y\leqq b_1\end{pmatrix},\\ 0, & (\text{ailleurs}),\end{cases}$$

et considérons la transformée de Fourier de cette fonction $f(x,y)$, c'est-à-dire

$$F(u,v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i(ux+vy)}dxdy$$
$$=\frac{1}{2\pi}\int_a^{a_1}\int_b^{b_1}f_1(x,y)e^{-i(ux+vy)}dxdy.$$

Il est alors aisé de voir que

$$|F(u,v)| \leqslant \frac{1}{2\pi |u|^m |v|^n} \int_a^{a_1}\int_b^{b_1} \times |f_1^{(m+n)}(x,y)|\,dxdy \leqslant A\frac{M_{m,n}}{|u|^m|v|^n},$$

d'où l'on tire

$$|F(u,v)| \leqslant \min_{m,n\geq 0} \frac{AM_{m,n}}{|u|^m|v|^n} = \frac{A}{T(|u|,|v|)} \leqslant Ae^{-\phi(|u|,|v|)}$$

En remarquant que

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)u^m v^n|\,dudv \leqslant AM_{m,n} \qquad \begin{pmatrix} m=0,1,2,\cdots \\ n=0,1,2,\cdots \end{pmatrix},$$

on voit que la transformée de Fourier de $F(u,v)$, c'est-à-dire

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i(xu+yv)}dudv$$

est indéfiniment dérivable et nulle sauf le domaine $\begin{pmatrix} a, a_1 \\ b, b_1 \end{pmatrix}$.

Nous pouvons donc énoncer le théorème suivant[1]:

Soit $\phi(u,v)$ une fonction positive, dérivable dans le domaine $\begin{pmatrix} 0, \infty \\ 0, \infty \end{pmatrix}$ et satisfaisant, pour chaque valeur positive d'un paramètre ν, à la condition $u\frac{d}{du}\phi(u,\nu u) \uparrow \infty$ avec $u \to \infty$. Soit C une classe de fonctions $f(x,y)$ appartenant à la classe $L^{*p}\begin{pmatrix} -\infty, \infty \\ -\infty, \infty \end{pmatrix}$ $[1 \leqslant p \leqslant \infty]$, indéfiniment dérivables dans ce domaine, et satisfaisant à la condition suivante:

$$|F(u,v)| \leqslant Ae^{-\phi(|u|,|v|)},$$

où $F(u,v)$ est la transformée de Fourier de la fonction $f(x,y)$. Alors la condition nécessaire et suffisante pour que la classe C soit quasi analytique est que l'intégrale double (18.3) diverge.

20. Nous pouvons immédiatement en déduire le corollaire suivant:

Soit $\phi(u,v)$ une fonction satisfaisant aux conditions du dernier théorème, et soit $f(x,y)$ une fonction appartenant à la classe $L^{*p}\begin{pmatrix} -\infty, \infty \\ -\infty, \infty \end{pmatrix}$ et satisfaisant à la condition suivante

$$|f(x,y)| \leqslant Ae^{-\phi(|x|,|y|)} \qquad (\text{où } A \text{ est une constante}).$$

Alors la divergence de l'intégrale double

$$(20.1) \qquad \int_1^{\infty}\int_1^{\infty} \phi(x,y)\frac{dx}{x^2}\frac{dy}{y^2}$$

est la condition nécessaire et suffisante pour que les relations

---

1. On voit ainsi que ce théorème étend, dans un certain sens, notre théorème dans la classe L^2 déjà exposé dans mon Mémoire (III). Mais tous les deux sont utilisables.

$$(20.2)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)x^m y^n dxdy = 0 \qquad \begin{pmatrix} m=0,\,1,\,2,\ldots \\ n=0,\,1,\,2,\ldots \end{pmatrix}$$

impliquent que la fonction $f(x,y)$ est identiquement nulle.

Il est d'abord clair que la condition est suffisante en vertu du théorème précédent. Ensuite, la condition est aussi nécessaire. Soit, en effet, $F(u,v)$ la transformée de Fourier de la fonction $f(x,y)$. On voit alors, en vertu du théorème fondamental du $n°18$, que $F(u,v)\in L^{*p}$ parce que $f(x,y)\in L^{*p}$ par hypothèse. Et la transformée de Fourier de cette fonction est $f(x,y)$ elle-même, qui satisfait à la condition $|f(x,y)|\in Ae^{-\phi(|x|,\,|y|)}$ par hypothèse. Or on a, grâce à (20.1),

$$F^{(m+n)}(0,0)=\frac{i^{m+n}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)x^m y^n\,dxdy=0 \qquad \begin{pmatrix} m=0,\,1,\,2,\ldots \\ n=0,\,1,\,2,\ldots \end{pmatrix},$$

ce qui prouve que $F(u,v)$ est identiquement nulle, et donc $f(x,y)$ l'est aussi. Notre corollaire est ainsi démontré.

§ 2. Antre forme du théorème du $n°19$

21. Démontrons maintenant le théorème suivant :

La conclusion du théorème du $n°19$ est aussi valable quand les conditions

$$|F(u,v)|\leqslant Ae^{-\phi(u,\,v)} \text{ et } \int_1^{\infty}\int_1^{\infty}\frac{\phi(u,v)}{u^2v^2}dudv=\infty$$

sont remplacées par

$$(21.1)\qquad \int_1^{\infty}\int_1^{\infty}\frac{du}{u^2}\frac{dv}{v^2}\log\int_u^{\infty}\int_v^{\infty}|F(\xi,\eta)|\,d\xi d\eta=-\infty.$$

Pour la démonstration on doit remarquer le lemme suivant :

Soit $f(x,y)$ une fonction appartenant à la classe L^{*p} $[1\leqslant p\leqslant\infty]$, et soit $F(u,v)$ la transformée de Fourier de $f(x,y)$. Si la fonction $f(x,y)$ est nulle presque partout dans un certain domaine, alors on peut toujours trouver une fonction $f_1(x,y)$ non équivalente à nul sans que $f(x,y)$ le soit, mais nulle dans un rectangle $\begin{pmatrix} -2U\leqslant x\leqslant 2U \\ -2V\leqslant y\leqslant 2V \end{pmatrix}$, et satisfaisant à la condition

$$(21.2)\qquad |f_1(x,y)|\leqslant\frac{N}{(1+x^2)(1+y^2)},$$

où N est une constante positive finie, avec sa transformée de Fourier vérifiant la condition

$$(21.3)\qquad |F_1(u,v)|\leqslant\frac{N}{(1+u^2)(1+v^2)}.$$

Pour affirmer ce lemme, soit la fonction $f(x,y)$ nulle dans un rectangle dont les côtés sont des longueurs $4U$ et $4V$. Alors il est clair que la fonction

$$(21.4)\qquad g(x,y)=\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|\xi|}{U}\right)\left(1-\frac{|\eta|}{V}\right)f(x+\xi,y+\eta)d\xi d\eta$$

est nulle dans le rectangle $\begin{pmatrix}-U,\ U\\ -V,\ V\end{pmatrix}$. Or on a

$$\frac{1}{2\pi}\int_{-U}^{U}\int_{-V}^{V}\left(1-\frac{|x|}{U}\right)\left(1-\frac{|y|}{V}\right)e^{-i(ux+vy)}dxdy$$
$$=\frac{8}{\pi UV}\frac{\sin^2\frac{1}{2}Uu}{u^2}\cdot\frac{\sin^2\frac{1}{2}Vv}{v^2}.$$

D'où l'on tire que la transformée de Fourier de la fonction $g(x,y)$ peut s'écrire sous la forme

$$(21.5)\qquad G(u,v)=\frac{8}{\pi UV}F(u,v)\frac{\sin^2\frac{1}{2}Uu}{u^2}\frac{\sin^2\frac{1}{2}Vv}{v^2}.$$

En posant donc

$$f_1(x,y)=4g(x,y)\frac{\sin^2\frac{1}{2}x}{x^2}\frac{\sin^2\frac{1}{2}y}{y^2},$$

on voit que la transformée de Fourier de cette fonction $f_1(x,y)$ peut s'écrire sous la forme

$$(21.6)\qquad F_1(u,v)=\int_{u-1}^{u+1}\int_{v-1}^{v+1}G(\xi,\eta)\{1-|u-\xi|\}\{1-|v-\eta|\}d\xi d\eta.$$

On voit donc, d'une part, que la condition (21.2) est certainement satisfaite de la définition même de la fonction $f_1(x,y)$, et d'autre part, que la condition (21.3) est aussi satisfaite, grâce aux (21.6) et (21.5); car, on a

$$(21.7)\qquad |F_1(u,v)|\leqslant\frac{N}{(1+u^2)(1+v^2)}\int_{u-1}^{u+1}\int_{v-1}^{v+1}|F(\xi,\eta)|d\xi d\eta$$
$$<\frac{N}{(1+u^2)(1+v^2)}\left[\int_{u-1}^{u+1}\int_{v-1}^{v+1}|F(\xi,\eta)|^{\frac{p}{p-1}}d\xi d\eta\right]^{\frac{p-1}{p}}\qquad[p>1]$$
$$<\frac{N}{(1+u^2)(1+v^2)}.$$

Il est à remarquer ici que le résultat (21.3) pour le cas où $p=1$ est aussi atteint de la borne uniforme de la fonction $F(u,v)$.

Comme il vient d'être dit à propos de la fonction $g(x,y)$, la fonction $f_1(x,y)$ est nulle dans le rectangle $\begin{pmatrix}-U,\ U\\ -V,\ V\end{pmatrix}$. Donc cette fonction $f_1(x,y)$ satisfait à toutes les conditions énoncées dans notre lemme, excepté qu'elle n'est pas nécessairement nulle dans le rectangle de centre d'origine. Or cette réquisition est atteinte par une transformation

simple, et $F_1(u, v)$ est seulement multipliée par un facteur de la forme $e^{i(uK+vL)}$, d'où il résulte que la codition (21.3) est aussi satisfaite. Notre lemme est aussi démontré.

22. Passons ensuite à la démonstration du théorème énoncé dans le dernier numéro. Supposons que la fonction $f(x, y)$ est nulle dans un domaine mais non identiquement nulle dans tout domaine $\begin{pmatrix} -\infty, \infty \\ -\infty, \infty \end{pmatrix}$. Alors il y a, en vertu du lemme précédent, une fonction $f_1(x, y)$ non équivalente à nulle, mais nulle dans un rectangle de centre d'origine, avec sa transformée de Fourier $F_1(u, v)$ vérifiant la relation

$$|F_1(u, v)| < \frac{N}{(1+u^2)(1+v^2)}.$$

Si donc on transforme le second membre de l'égalité

$$f_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(u, v)e^{i(xu+yv)}dudv$$

par l'intégration par parties, on a

$$\frac{f_1(x, y)}{i^2xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(xu+yv)}dudv\int_{u}^{\infty}\int_{v}^{\infty} F_1(\xi, \eta)d\xi d\eta.$$

On voit donc que la fonction $\frac{f_1(x, y)}{i^2xy}$ est identiquement nulle dans un domaine, et de plus, que sa transformée de Fourier peut s'écrire sous la forme

$$F_2(u, v) = \int_{u}^{\infty}\int_{v}^{\infty} F_1(\xi, \eta)d\xi d\eta.$$

Et l'on a, en vertu de (21.7),

$$|F_2(u, v)| < A\int_{u-1}^{\infty}\int_{v-1}^{\infty} |F(\xi, \eta)|\, d\xi d\eta.$$

En posant donc

$$\phi(u, v) = -\log\int_{u-1}^{\infty}\int_{v-1}^{\infty} |F(\xi, \eta)|\, d\xi d\eta,$$

on a immédiatement, grâce à (21.1),

$$\int_{2}^{\infty}\int_{2}^{\infty}\phi(u, v)\frac{du}{u^2}\frac{dv}{v^2} = \infty.$$

Or on a

$$|F(u, v)| < Ae^{-\phi(u, v)} \qquad \begin{pmatrix} u \geqq 2 \\ v \geqq 2 \end{pmatrix}.$$

On voit donc, en vertu du théorème du n° 19, que la fonction $\frac{f_1(x, y)}{i^2xy}$ est identiquement nulle, et donc la fonction $f_1(x, y)$ l'est aussi.

Notre théorème est donc démontré.

Je suis heureux de pouvoir exprimer mon profond remerciement au professeur M. Tosizô Matumoto, au Maître qui a bien voulu lire mon manuscrit et qui m'a fait quelques remarques très utiles.

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